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**AN OPTIMIZATION TECHNIQUE FOR
PULSE WIDTH MODULATED SYSTEMS**

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ABSTRACT

A procedure for the design of a controller to optimize a certain class of pulse width modulated systems is presented. The process to be controlled is time-invariant, of arbitrary order and excited by a sequence of pulses generated from information available at arbitrary sampling instants. Input information to the system is quite general and includes random and deterministic phenomena. Identification of the plant is accomplished using state variable notation and linear estimation techniques. Prediction of the future plant behavior is also performed with these techniques and the controller is designed to optimize the predicted plant performance by minimizing a measure of the future system errors. The mechanized optimal control law or controller program, develops the pulse to be applied at any sampling instant by specifying the pulse width and associated sign. The controller is not adaptive in the sense of redesign occurring as new information becomes available from the estimator. It does, however, accomplish the goal of optimization by deciding the form of the pulse width actuating signal utilizing the future errors of the system.

The general results are applied to several examples through digital computer simulation. The optimal pulse width controller is shown to produce far better performance than normal pulse width control which utilizes present errors to develop the actuating signal to be applied at that same instant and for some time in the future. The results of considering a family of step function inputs to the system are shown for a third and fourth order plant.

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1. INTRODUCTION

1.1 Pulse Width Control

Pulse width modulated control systems have been in existence for more than 65 years¹ and only recently, activity and interest have increased in this area. The present frequent use of pulse width control has developed in spite of the almost complete lack of elegant mathematical tools for analysis and synthesis since the description of these systems requires nonlinear differential or difference equations. This seems to indicate the possibility that this type of control is so effective and natural in a large class of control problems that defiance of analysis is not sufficient justification to avoid its use.

Pulse width control is a generalized form of relay or on-off control which provides a finer, more precise response. The advantages arise mainly from the ability to regulate the steady state ripple oscillation frequency, to obtain improved accuracy due to elimination of dead zone, and to include possible time sharing of the control computer.

Pulse width modulated control has direct application to satellite and space vehicle attitude control. In many cases this situation requires power to be modulated in an on-off fashion where a control computer must be time shared leaving very little choice other than to use pulse width control. Recent experimental efforts in this area have shown that time dependent switching techniques provide precision attitude control with low thrust vapor jets achieving results not possible with conventional on-off methods^{2,3}.

Further investigation of pulse width control is justifiable on the basis of attitude control applications alone, but let us consider for a moment, another system utilizing a more general form of pulsed control.

Man is the ultimate control system. Many physical devices have been modeled after a particular human function, and today we find an ever increasing effort toward the understanding of biological functions so that related systems may be improved and extended. The method of information transmission in the human being is a combination of pulse width and pulse repetition modulation. We find cardiac pulsatory phenomena remarkable in all senses⁴, particularly in accuracy and reliability of control, since it is literally vital. More incredible is the completely integrated pulse communication network in the nervous system^{5,6}.

The sensory receptors provide the pulsed form of physical stimuli that excite our system achieving extreme sensitivities (e.g. the retina of the eye). From these considerations it might be concluded that pulse modulation would be the ideal method of information transmission for many control purposes. In any case, future investigation of this type and simpler types of pulsed control is certainly justifiable.

Several authors, R.F. Nease⁷, R.E. Andeen⁸, T.T. Kadota⁹, E. Polak^{10,11,12,13}, T. Nishimura and E.I. Jury¹⁴, I.V. Pyshkin^{15,16}, F.R. Delfeld and G.J. Murphy¹⁷, S.C. Gupta and E.I. Jury¹⁸, S.C. Gupta¹⁹, have published efforts in limit cycles, finite duration processes, analytical techniques and optimization. The techniques for optimization have been limited to second order regulator systems to date. The purpose of this work is to develop an optimization technique applicable to controlled processes of any order when the system is subjected to a broad class of possible input phenomena.

1.2 Statement of the Problem

The system to be investigated in this report is assumed quite general in form. Restrictions on the nature of the system which would degenerate usefulness of the optimization technique to only academic problems are avoided. Practical considerations have taken precedence in the formulation of this problem where ever possible.

A time-invariant linear system called the controlled system or plant is assumed excited by a pulse width modulated control signal. A feedback controller is to be designed to provide the best control or actuating signal utilizing information about the state of the system at only arbitrary sample instants. The information about the state of the system is constrained in that only noisy measurements are available for estimation purposes. The system is to be optimum for this pulse width modulated control signal where optimum is used in the sense of minimizing a measure of system error.

The mechanized control law will constitute the design of the optimal controller and the solution to the problem. The controller is not to be adaptive in the sense of redesign occurring as new information becomes available about the state of the system. It is to provide a program of logical steps that will be executed during each sampling interval determining the exact form of the pulsed actuating signal by specifying the pulse width and associa-

The command inputs to the system are quite general in form in that they include both random and deterministic phenomena.

A block diagram of the system under investigation in this report is shown in Figure 1.

1.3 Optimal Control Law

The optimization of a class of pulse width modulated systems with linear plants was first considered by W.L. Nelson²⁰. This class consisted of regulator systems and is a problem of minimal time control. Nelson's approach follows a general technique for the minimal time control of pulse amplitude modulated systems with saturation as presented by R.E. Kalman²¹. This procedure consists of dividing the state space into regions for which the system may be taken to the desired state in a minimal number of sampling periods, establishing a canonical vector representation for initial states in the respective regions. The approach is limited to plants of second order due to the conceptual difficulty of determining the optimal control regions in higher dimensional state spaces.

Nelson did not, however, present a procedure for constructing canonical vector representations for arbitrary initial states. E. Polak^{11,12,13} succeeded in accomplishing this following a method suggested by C.A. Desoer and J. Wing²² on minimal time control of pulse amplitude modulated systems with saturation. Polak was able to solve the regulator problem for some second order linear and nonlinear plants. The limitation to second order is again prominent due to conceptual difficulty in higher dimensional state spaces.

The previous efforts have been based on the approaches used to optimize pulse amplitude modulated systems. The extension of techniques applicable to pulse amplitude systems may itself be a fundamental limitation and thus not worthy of further investigation. Considering pulse width control as a fundamentally different type of control with its own intrinsic properties, an approach to the problem may be found. Variational techniques have been attempted with little success²³. Combinatorial techniques as previously applied seem to be the only other alternative. Dynamic programming as an organized logical scheme is directly related to the sampled system and seems to be the most likely candidate here. Dynamic programming then, will provide the logical framework for the optimization scheme where the pulse width modulated control

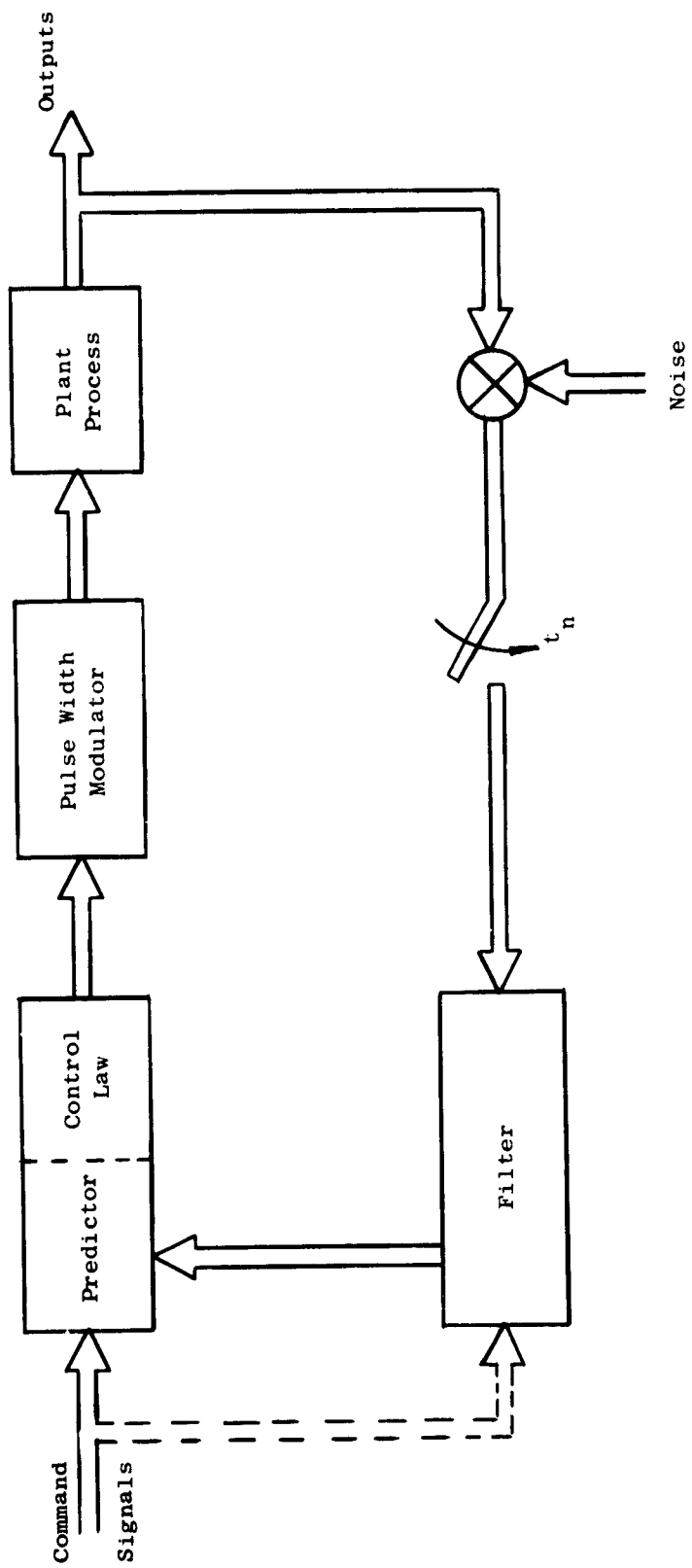


Figure 1. Block Diagram of the System

signal will be treated as a part of the system constraint rather than as a separate constraint.

The previous contributions in the application of dynamic programming have been formulated for time varying plants. The first such application to a time varying plant was presented by R.E. Kalman and R.W. Koepcke²⁴. Later, T.L. Gunckel²⁵ and, independently, P.D. Joseph and J.T. Tou²⁶, reported results in the control of linear processes subject to multiplicative as well as additive random effects. A very practical extension to include random multiplicative phenomena which are correlated both in time and with each other, and not restricted to independence from instant to instant, has been presented by C. Pottle²⁷.

The system under study here is not assumed time varying for simplification, but is approached in exactly the same fashion as mentioned above. The additional constraint of pulse width modulation is included and the information about the state at sampling instants is also constrained. Minimization of a measure of predicted system errors on a step by step basis provides the final necessary ingredient for the solution to the optimization problem.

Estimation of the state and prediction of the future system error is accomplished by linear estimation techniques in state vector notation as presented by R.E. Kalman^{28,29}.

2. DESCRIPTION OF THE SYSTEM

2.1 General

A mathematical description of the system is necessary. We must decide on a general model which is sufficiently complex to represent the behavior of a large class of physical processes with a high degree of accuracy. The model must also be simple so that useful results may be obtained easily. Any model under consideration is also required to represent linear plants subject to pulse width information.

The representation of a pulse using transform techniques where the width is a complex function of system responses seems entirely inadequate. Thus, a time domain representation will be used.

2.2 State Vector Representation

A general model which can represent a system with finitely many degrees of freedom is one using the "state space" concept introduced by Kalman and Bertram³⁰. The state of the system at any instant of time is represented by a vector whose components are called the "state variables." This vector defines a point in the state space. The dimension of the state vector is the smallest possibly to completely describe the behavior of the system at that point in time.

For a general time invariant plant, its dynamic behavior is assumed to be adequately approximated by the following vector equation.

$$\begin{aligned}\dot{\bar{x}}(t) &= A \bar{x}(t) + B \bar{u}(t) \\ \bar{c}(t) &= M^x \bar{x}(t)\end{aligned}\tag{2.1}$$

where

$\bar{x}(t)$ is an (mx1) state vector
 $\bar{c}(t)$ is a (px1) output vector
 $\bar{u}(t)$ is a (qx1) input vector
 A is an (mxm) transition matrix
 B is an (mxq) distribution matrix
 M^x is a (pxq) output matrix

If the system is time invariant, the transition, distribution and output matrices have elements not dependent on time.

The actuating or control signal is $\bar{u}(t)$ and will have a form dictated by the constraints imposed by the problem. For the particular problem at hand, that of pulse width modulated systems, the control signal must then satisfy these requirements. The i th component of $\bar{u}(t)$ is

$$u_i(t) = \sum_{n=0}^{\infty} [u_1(t-t_n) - u_1(t-t_n - \rho_{ni})] \epsilon_{ni}$$

where $u_1(t)$ is the unit step function and ρ_{ni} is the pulse width of the n th interval for the i th channel and ϵ_{ni} is the associated sign (± 1). Thus, we see that the pulsed control signal is described by the sequence of pairs of vectors $(\bar{\rho}_n, \bar{\epsilon}_n)$. The solution to the optimization problem is the generation of a sequence of couples $(\bar{\rho}_n, \bar{\epsilon}_n)$ determined in such a way that the performance of the system satisfies the condition for optimality.

For the control vector $\bar{u}(t)$, we have

$$\bar{u}(t) = \sum_{n=0}^{\infty} \begin{bmatrix} [u_1(t-t_n) - u_1(t-t_n - \rho_{n1})] \epsilon_{n1} \\ \vdots \\ [u_1(t-t_n) - u_1(t-t_n - \rho_{nq})] \epsilon_{nq} \end{bmatrix}$$

where

$\bar{\rho}_n$ is a $(qx1)$ pulse width vector

$\bar{\epsilon}_n$ is a $(qx1)$ sign vector

Defining a unit step function matrix will simplify the description of the control vector in general form. Let

$$U_1(t) = I u(t)$$

where I is a (qxq) unit matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & & & 0 \\ . & & & & \\ 0 & 0 & & & 1 \end{bmatrix}$$

And, let $U_1(\bar{a})$, where \bar{a} is a $(q \times 1)$ vector, be defined as

$$U_1(\bar{a}) = \begin{bmatrix} u_1(a_1) & 0 & 0 & \dots & 0 \\ 0 & u_1(a_2) & & \dots & 0 \\ . & & & & \\ . & & & & \\ 0 & 0 & & \dots & u_1(a_q) \end{bmatrix}$$

Then, the control vector may be written

$$\bar{u}(t) = \sum_{n=0}^{\infty} \left[U_1(t-t_n) - U_1(t-t_n - \bar{\rho}_n) \right] \bar{\epsilon}_n \quad (2.2)$$

Using this notation, we may now determine the discrete form of the state vector representation for the system at the sampling instants. The solution for the state vector in equation (2.1) is

$$\bar{x}(t) = \phi^x(t, t_n) \bar{x}(t_n) + \int_{t_n}^t \phi^x(t, \lambda) B \bar{u}(\lambda) d\lambda \quad \bar{\epsilon}_n \quad (2.3)$$

where $\phi^x(t, t_n) = e^{A(t-t_n)}$ and is called the characteristic matrix.

Using the expression developed for the control signal $\bar{u}(t)$, equation (2.3) becomes

$$\bar{x}(t) = \phi^x(t, t_n) \bar{x}(t_n) + \left[\int_{t_n}^t \phi(t, \lambda) B d\lambda - \int_{t_n}^t \phi^x(t, \lambda) B U_1(\lambda - t_n - \bar{\rho}_n) d\lambda \right] \bar{\epsilon}_n$$

for $t_n \leq t \leq t_{n+1}$.

To simplify notation define

$$\int_a^{(\bar{b})_i} H dp = \begin{bmatrix} \int_a^{b_1} h_{11} dp & \dots & \int_a^{b_m} h_{1m} dp \\ \vdots & & \vdots \\ \int_a^{b_1} h_{q1} dp & \dots & \int_a^{b_m} h_{qm} dp \end{bmatrix}$$

where $H = (h_{ij})$ $i = 1, 2, \dots, q$ $j = 1, 2, \dots, m$ and \bar{b} is a $(q \times 1)$ vector $= (b_k)$ $k = 1, 2, \dots, q$.

The expression for $\bar{x}(t)$ may be written now as

$$\bar{x}(t) = \phi^x(t, t_n) \bar{x}(t_n) + \left[\int_{t_n}^t \phi^x(t, \lambda) B d\lambda - \int_{t_n + (\bar{\rho}_n)_i}^t \phi^x(t, \lambda) B U_1 (\lambda - t_n - \bar{\rho}_n) d\lambda \right] \bar{\epsilon}_n \quad (2.4)$$

$i = 1, 2, \dots, q$

For $t_n + (\bar{\rho}_n)_i \leq t \leq t_{n+1}$ $i = 1, 2, \dots, q$ Equation (2.4) may be simplified by combining the two integrals.

$$\bar{x}(t) = \phi^x(t, t_n) \bar{x}(t_n) + \int_{t_n}^{t_n + (\bar{\rho}_n)_i} \phi^x(t, \lambda) B d\lambda \bar{\epsilon}_n \quad (2.5)$$

$$i = 1, 2, \dots, q$$

Changing the variable of integration in (2.5)

$$\bar{x}(t) = \phi^x(t, t_n) \bar{x}(t_n) + \int_0^{(\bar{\rho}_n)_i} e^{A(t-t_n-p)} B dp \bar{\epsilon}_n \quad (2.6)$$

and

$$\bar{x}(t) = \phi^x(t, t_n) \bar{x}(t_n) + \phi^x(t, t_n) \int_0^{(\bar{\rho}_n)_i} e^{-Ap} B dp \bar{\epsilon}_n \quad (2.7)$$

$$\text{for } i = 1, 2, \dots, q \quad t_n + (\bar{\rho}_n)_i \leq t \leq t_{n+1}.$$

For the response at the sampling instants, $t = t_{n+1}$ and equation (2.7) becomes

$$\bar{x}(t_{n+1}) = \phi^x(t_{n+1}, t_n) \bar{x}(t_n) + \phi^x(t_{n+1}, t_n) \int_0^{\bar{p}_n)_i} e^{-A p_{Bd} p} \bar{\epsilon}_n \quad (2.8)$$

Letting $\bar{x}(t_n) = \bar{x}_n$ and $\phi^x(t_{n+1}, t_n) = \phi_n^x$, equation (2.8) becomes

$$\bar{x}_{n+1} = \phi_n^x \bar{x}_n + \phi_n^x \int_0^{\bar{p}_n)_i} e^{-A \lambda} B d\lambda \bar{\epsilon}_n \quad (2.9)$$

The controlled system is thus adequately approximated at the sampling instants by the vector equations

$$\begin{aligned} \bar{x}_{n+1} &= \phi_n^x \bar{x}_n + A_n \bar{\epsilon}_n \\ \bar{c}_n &= M_n^x \bar{x}_n \end{aligned} \quad (2.10)$$

where

$$A_n = \phi_n^x \int_0^{\bar{p}_n)_i} e^{-A \lambda} B d\lambda \quad i=1, 2, \dots, q \quad (2.11)$$

The equations (2.10) describe what will be called the plant process at the sampling instants. The state of the system at the sampling instants is controlled by the choice of the adjustable parameters in the control signal which are the pulse widths \bar{p}_n and signs $\bar{\epsilon}_n$. The control law will provide the technique by which this pair $(\bar{p}_n, \bar{\epsilon}_n)$ may be determined in the desired manner.

2.3 The Input Process

Kalman has introduced the notion of a reference input vector $\bar{r}(t_n)$ which can be regarded as being generated as the output of a linear dynamic system. Following this idea, the input or reference signal $\bar{r}(t_n)$ is assumed to be defined by an input process. That is, the reference input vector representing the desired value of the plant output vector is adequately approximated, at the sampling instants, by the following vector difference equation.

$$\begin{aligned}\bar{w}_{n+1} &= \phi_n^w \bar{w}_n + \Delta_n^w \bar{s}_n \\ \bar{r}_n &= M_n^w \bar{w}_n\end{aligned}\tag{2.12}$$

where

\bar{w}_n is a (vxl) state vector
 \bar{r}_n is a (pxl) reference input vector
 \bar{s}_n is an (hxl) process input vector
 ϕ_n^w is a (vxv) transition matrix
 Δ_n^w is a (vxh) distribution matrix
 M_n^w is a (pxv) process output matrix

The dimensionality of the reference input vector \bar{r}_n has been assumed the same as that of the output vector \bar{c}_n . The components of the process input vector are assumed to be random variables with zero mean and to be statistically independent from one sampling instant to the next and independent of the transition and distribution matrices. Thus

$$E(\bar{s}_n) = 0 \quad E(\bar{s}_n^t \bar{s}_m) = \begin{cases} 0 & n \neq m \\ I & n = m \end{cases}\tag{2.13}$$

Note that this model is sufficiently general in that it includes

- a) The regulator problem when $\bar{r}_n = 0$ for all n
- b) The deterministic problem when $\Delta_n^w = 0$ for all n
- c) The random input problem when $\bar{r}_0 = 0$

Also, a combination of these classes of inputs can be accommodated by the model using suitably defined transition and distribution matrices.

The assumed input vector $\bar{r}(t)$ may or may not belong or be a subset of the actual command input vector $\bar{r}'(t)$. If it is a subset, the system will be optimum with respect to \bar{r}'_n ; if not, only an approximation will be obtained, and an

approximation at the sampling instants. The designer must decide on a class $\{\bar{r}_n\}$ to which his system is to be optimum, and then determine the necessary matrices since the controller will depend on these quantities.

The requirement that the reference inputs be describable as the output of some process limits the class of possible inputs, but this restriction is not too severe for our purpose.

2.4 The System and Information Constraints

Following the suggestion of Kalman and Koepke²², the equations for the plant process and the input process are combined into a single equation. This is accomplished as follows:

$$\begin{bmatrix} \bar{x}_{n+1} \\ \bar{w}_{n+1} \end{bmatrix} = \begin{bmatrix} \phi_n^x & 0 \\ 0 & \phi_n^w \end{bmatrix} \begin{bmatrix} \bar{x}_n \\ \bar{w}_n \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta_n^w \end{bmatrix} \bar{s}_n + \begin{bmatrix} A_n \\ 0 \end{bmatrix} \bar{\epsilon}_n$$

$$\begin{bmatrix} \bar{c}_n \\ \bar{r}_n \end{bmatrix} = \begin{bmatrix} M_n^x & 0 \\ 0 & M_n^w \end{bmatrix} \begin{bmatrix} \bar{x}_n \\ \bar{w}_n \end{bmatrix} \quad (2.14)$$

Defining

$$\bar{z}_n = \begin{bmatrix} \bar{x}_n \\ \bar{w}_n \end{bmatrix} \quad \bar{y}_n = \begin{bmatrix} \bar{c}_n \\ \bar{r}_n \end{bmatrix} \quad M_n = \begin{bmatrix} M_n^x & 0 \\ 0 & M_n^w \end{bmatrix}$$

$$\phi_n = \begin{bmatrix} \phi_n^x & 0 \\ 0 & \phi_n^w \end{bmatrix} \quad \Delta_n = \begin{bmatrix} 0 \\ \Delta_n^w \end{bmatrix} \quad \bar{A}_n = \begin{bmatrix} A_n \\ 0 \end{bmatrix}$$

we may write (2.14) as

$$\bar{z}_{n+1} = \phi_n \bar{z}_n + \Delta_n \bar{s}_n + \bar{A}_n \bar{\epsilon}_n$$

$$\bar{y}_n = M_n \bar{z}_n \quad (2.15)$$

where

\bar{z}_n is an $(m+v \times 1)$ state vector

\bar{y}_n is a $(2p \times 1)$ output vector

$\bar{\epsilon}_n$ is a $(q \times 1)$ sign vector

\bar{s}_n is an $(h \times 1)$ process input vector

ϕ_n is an $(m+v \times m+v)$ transition matrix

Δ_n is an $(m+v \times h)$ distribution matrix

\bar{A}_n is an $(m+v \times q)$ distribution matrix

M_n is a $(2p \times m+v)$ output matrix

The complete set of equations (2.15) which include the plant process and the input process will be called the system constraint.

The remaining constraint is concerned with the amount of information available about the state of the system at any particular sampling instant. If the special case of exact measurement of the state exists, then we have no restriction on information and no need to consider the limitation. The usual case, however, is the one where only certain measurements are available and these measurements together with past history constitute the data available for estimation of the state. Since the existing data in this case is limited by additive noise, for example, we speak of the information constraint.

3. OPTIMUM PLANT CONTROL

3.1 General

We are interested in finding an "optimum" control law for the pulse width modulated system described previously. The sense of the word "optimum" is defined by the introduction of a performance index which provides a measure of the system behavior. The method of dynamic programming is then used to derive a control law for the processes of the type introduced in Chapter 2. Actually, the method is applicable to many cases of the general model with varied constraints, but here we will examine only one particular class of problems, namely that of pulse width modulated systems.

3.2 The Performance Index

An optimum control system is characterized by a performance index which is a function of the system variables and parameters. For extremal values of the scalar function, the system is said to be optimum in this sense, but obviously, its optimality is subject to the performance index chosen. This choice of a performance index is thus, perhaps the most important decision to be made. Unfortunately, there is really no way a designer may specify what he wants in the performance of a particular system so that it is optimum in any absolute sense³¹. We are thus faced with choosing a performance index which is convenient to work with mathematically and which also corresponds to some reasonable definition of "desirable behavior."

The measure of system performance which seems suitable in general would be the weighted error of the system. This error in the system is the difference between the actual plant output and the desired output. It is assumed that a quadratic combination of the error vector will be a suitable measure of performance.

$$E_n = (\bar{c}_n - \bar{r}_n)^T Q'_n (\bar{c}_n - \bar{r}_n) \quad (3.1)$$

The matrix Q'_n is taken to be symmetric and positive definite and simply corresponds to the desired weighting of the errors in the system. Thus, the single number E_n represents the performance of the system at the nth instant of time.

The measure of performance may be rewritten in terms of the state vector \bar{z}_n .

Since

$$(\bar{c}_n - \bar{r}_n) = \begin{bmatrix} M_n^x & -M_n^w \end{bmatrix} \bar{z}_n ,$$

then

$$E_n = (\bar{c}_n - \bar{r}_n)^T Q'_n (\bar{c}_n - \bar{r}_n)$$

or

$$E_n = \bar{z}_n^T \begin{bmatrix} M_n^{xT} \\ -M_n^{wT} \end{bmatrix} Q'_n \begin{bmatrix} M_n^x & -M_n^w \end{bmatrix} \bar{z}_n \quad (3.2)$$

Defining a new weighting matrix Q_n

$$Q_n = \begin{bmatrix} M_n^{xT} \\ -M_n^{wT} \end{bmatrix} Q'_n \begin{bmatrix} M_n^x & -M_n^w \end{bmatrix} \quad (3.3)$$

we have

$$E_n = \bar{z}_n^T Q_n \bar{z}_n \quad (3.4)$$

for the measure of system performance.

For operation of the system involving N steps, a reasonable performance index is the sum of the individual performance measures E_n .

Define the performance index as J_{N-n+1}

$$J_{N-n+1} = \sum_{i=n}^N \bar{z}_i^T Q_i \bar{z}_i . \quad (3.5)$$

The performance index is thus the sum of the weighted mean square errors at the sampling instants.

3.3 Criterion for Optimal Performance

To obtain optimum pulse width control, we will minimize the performance index by generating the sequence of both the width and sign of the pulse at each sampling instant. The final sequence of pairs and the control law to obtain them will constitute the solution to the optimization problem. Optimum is used here in the sense of minimizing this quadratic function of system error.

The precise statement of the optimization problem is thus the minimization of the performance index over the class of acceptable pairs of pulse width and sign, subject to system and information constraints.

That is

$$\begin{array}{l} \text{Minimize} \\ (\bar{\rho}_n, \bar{\epsilon}_n) \end{array} J_{N-n+1} \quad (3.6)$$

Note, however, that the above statement is restricted to only deterministic conditions. In the more general case, random variations in the system preclude the possibility of finding an input which will be optimum in every case. In other words, the performance index is also a random variable. The conditional expectation of the performance index will then be minimized. Let D_n represent the information available for the determination of the state of the system at time $t = t_n$. The optimization problem may be stated as follows:

$$\begin{array}{l} \text{Minimize} \quad E (J_{N-n+1} / D_n) \\ \{(\bar{\rho}_n, \bar{\epsilon}_n)\} \end{array}$$

where

$$J_{N-n+1} = \sum_{i=n}^N \bar{z}_i^T Q_i \bar{z}_i$$

subject to the system constraint

$$\bar{z}_{n+1} = \phi_n \bar{z}_n + \Delta_n \bar{s}_n + \bar{A}_n \bar{\epsilon}_n$$

$$\bar{Y}_n = M_n \bar{z}_n$$

and the information constraint

$$\bar{Y}'_n = \bar{Y}_n + \bar{\alpha}_n$$

$$D_n = \{\bar{Y}'_n, \bar{Y}'_{n-1}, \dots, \bar{Y}'_0\}$$

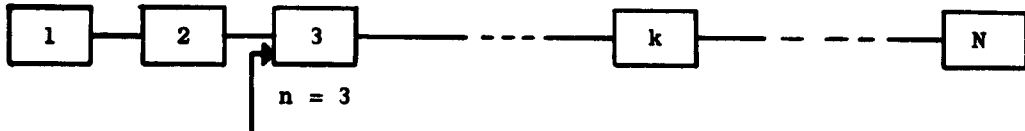
D_n is the set of measurements at time $t = t_{n-1}$ available for estimation of the state.

A loss function is defined as follows:

$$I_{N-n+1} = \begin{array}{l} \text{Minimum} \\ \{(\bar{\rho}_n, \bar{\epsilon}_n)\} \end{array} E (J_{N-n+1} / D_n) \quad (3.8)$$

In order to clarify the notation used above, consider a simple example of an N stage process.

Let N = the total number of stages in the process
 n = the number of the stage of interest.



When at stage n , the number of remaining stages which constitutes the new process is $N-n+1$.

3.4 Derivation of the Control Law

The minimization of the performance index over the class of possible pairs $(\bar{\phi}_n, \bar{\epsilon}_n)$ will provide the solution to the problem and the loss function provides the measure of achievement at each stage of the process. To determine the control law we will follow the framework of dynamic programming^{32,33} and derive the principle of optimality for this case since Bellman's principle of optimality does not apply directly when concerned with a conditional expectation.

Consider the loss function,

$$I_{N-n+1} = \underset{(\bar{\phi}_n, \bar{\epsilon}_n)}{\text{Minimum}} E(J_{N-n+1}/D_n) \quad (3.9)$$

or

$$I_{N-n+1} = \underset{(\bar{\phi}_n, \bar{\epsilon}_n)}{\text{Minimum}} E \sum_{i=n}^N \left[\bar{z}_i^T Q_i \bar{z}_i / D_n \right] \quad (3.10)$$

Taking the first term of the summation and writing it separately,

$$I_{N-n+1} = \underset{(\bar{\phi}_n, \bar{\epsilon}_n)}{\text{Minimum}} E \left[(\bar{z}_n^T Q_n \bar{z}_n + \sum_{i=n+1}^N \bar{z}_i^T Q_i \bar{z}_i) / D_n \right] \quad (3.11)$$

Writing the class of pairs for which the minimization is to be accomplished, we have

$$I_{N-n+1} = \underset{(\bar{\rho}_0, \bar{\epsilon}_0)}{\text{Min}} \underset{(\bar{\rho}_1, \bar{\epsilon}_1)}{\text{Min}} \dots \underset{(\bar{\rho}_{n-1}, \bar{\epsilon}_{n-1})}{\text{Min}} E \left[\bar{z}_n^T Q_n \bar{z}_n + \sum_{i=n+1}^N \bar{z}_i^T Q_i \bar{z}_i / D_{n-1} \right] \quad (3.12)$$

Examining the first term in the expression (3.12), we see that it is the performance measure of system error at time t_n . The sequence of pulses that have been applied before time t_n are initial decisions that have already been made. Therefore, the remaining decisions must constitute an optimal policy with regard to the state resulting from the application of the previous pulses. We may write then,

$$I_{N-n+1} = \underset{(\bar{\rho}_{n-1}, \bar{\epsilon}_{n-1})}{\text{Minimum}} E \left[(\bar{z}_n^T Q_n \bar{z}_n / D_{n-1}) + \underset{(\bar{\rho}_n, \bar{\epsilon}_n)}{\text{Min}} \underset{(\bar{\rho}_{n+1}, \bar{\epsilon}_{n+1})}{\text{Min}} \dots \underset{(\bar{\rho}_N, \bar{\epsilon}_N)}{\text{Min}} E \sum_{i=n+1}^N (z_i^T Q_i z_i / D_{n-1}) \right] \quad (3.13)$$

Note that the last term in the expression is I_{N-n} and write

$$I_{N-n+1} = \underset{(\bar{\rho}_{n-1}, \bar{\epsilon}_{n-1})}{\text{Minimum}} E \left[\bar{z}_n^T Q_n \bar{z}_n / D_{n-1} + I_{N-n} \right] \quad (3.14)$$

where $I_0 = 0$ and $n = 1, 2, \dots, N$.

The introduction of the effect of future errors is contained in the derivation above when the minimization terms are distributed throughout the expression and the one retained is $(\bar{\rho}_{n-1}, \bar{\epsilon}_{n-1})$. This pair corresponds to the pulse applied at time t_{n-1} and the performance measure at time t_n is minimized by this pair. Thus, the function of error one sample ahead of the time of application of the pulse is the future function of error that is minimized. The loss function I_{N-n+1} is the number representing the achievement of this pulse.

At this point, there certainly exists the possibility of looking ahead further in the system to minimize the error function perhaps at two sample

instants ahead. These results may easily be seen and we will not carry through the details of that particular derivation here.

Continuing with the solution of the recurrence relation involving the loss functions, we will first let $n=N$ and solve for a single stage process. This will accomplish the first step in the iterative procedure for the final solution. For $n = N$ equation (3.14) is

$$I_1 = (\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1}) \text{ Minimum } E (\bar{z}_N^T Q_N \bar{z}_N / D_{N-1}) \quad (3.15)$$

since $I_0 = 0$.

Introducing the system constraint by substitution for \bar{z}_N from equation (3.7),

$$I_1 = (\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1}) \text{ Minimum } E \left[(\phi_{N-1} \bar{z}_{N-1} + \Delta_{N-1} \bar{s}_{N-1} + \bar{A}_{N-1} \bar{\epsilon}_{N-1})^T Q_N \dots \right. \\ \left. \dots (\phi_{N-1} \bar{z}_{N-1} + \Delta_{N-1} \bar{s}_{N-1} + \bar{A}_{N-1} \bar{\epsilon}_{N-1}) / D_{N-1} \right] \quad (3.16)$$

Performing the indicated operations and expanding,

$$I_1 = (\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1}) \text{ Minimum } E \left[\bar{z}_{N-1}^T \phi_{N-1}^T Q_N \phi_{N-1} \bar{z}_{N-1} + \bar{z}_{N-1}^T \phi_{N-1}^T Q_N \Delta_{N-1} \bar{s}_{N-1} \right. \\ + \bar{z}_{N-1}^T \phi_{N-1}^T Q_N \bar{A}_{N-1} \bar{\epsilon}_{N-1} + \bar{s}_{N-1}^T \Delta_{N-1}^T Q_N \phi_{N-1} \bar{z}_{N-1} \\ + \bar{s}_{N-1}^T \Delta_{N-1}^T Q_N \Delta_{N-1} \bar{s}_{N-1} + \bar{s}_{N-1}^T \Delta_{N-1}^T Q_N \bar{A}_{N-1} \bar{\epsilon}_{N-1} \\ + \bar{\epsilon}_{N-1}^T \bar{A}_{N-1}^T Q_N \phi_{N-1} \bar{z}_{N-1} + \bar{\epsilon}_{N-1}^T \bar{A}_{N-1}^T Q_N \Delta_{N-1} \bar{s}_{N-1} \\ \left. + \bar{\epsilon}_{N-1}^T \bar{A}_{N-1}^T Q_N \bar{A}_{N-1} \bar{\epsilon}_{N-1} / D_{N-1} \right] \quad (3.17)$$

The first, third, seventh and last terms may be rewritten in simpler form using the following equivalence.

$$\begin{aligned}
& (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1})^T Q_N (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}) = \\
& = \bar{\epsilon}_{N-1}^T \bar{A}_{N-1}^T Q_N \bar{A}_{N-1} \bar{\epsilon}_{N-1} + \bar{z}_{N-1}^T \phi_{N-1}^T Q_N \phi_{N-1} \bar{z}_{N-1} + \\
& - \bar{\epsilon}_{N-1}^T \bar{A}_{N-1}^T Q_N \phi_{N-1} \bar{z}_{N-1} - \bar{z}_{N-1}^T \phi_{N-1}^T Q_N \bar{A}_{N-1} \bar{\epsilon}_{N-1}
\end{aligned} \tag{3.18}$$

Thus, equation (3.17) may be written as

$$\begin{aligned}
I_1 = & \underset{(\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1})}{\text{Minimum}} E \left[(\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1})^T Q_N (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}) + \right. \\
& + \bar{s}_{N-1}^T \Delta_{N-1}^T Q_N \Delta_{N-1} \bar{s}_{N-1} + \bar{z}_{N-1}^T \phi_{N-1}^T Q_N \Delta_{N-1} \bar{s}_{N-1} + \\
& + \bar{s}_{N-1}^T \Delta_{N-1}^T Q_N \phi_{N-1} \bar{z}_{N-1} + \bar{s}_{N-1}^T \Delta_{N-1}^T Q_N \bar{A}_{N-1} \bar{\epsilon}_{N-1} + \\
& \left. + \bar{\epsilon}_{N-1}^T \bar{A}_{N-1}^T Q_N \Delta_{N-1} \bar{s}_{N-1} / D_{N-1} \right]
\end{aligned} \tag{3.19}$$

If the input to the system is deterministic then $\Delta_n = 0$ for all n and we have only the first term in the expression (3.19). If the input is random with \bar{s}_n satisfying the assumed properties, then taking the expectation with respect to \bar{s}_n eliminates all but the first two terms.

Taking the expectation with respect to \bar{s} we then may write

$$\begin{aligned}
I_1 = & \underset{(\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1})}{\text{Minimum}} E \left[(\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1})^T Q_N \dots \right. \\
& \left. \dots (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}) + \gamma_0 / D_{N-1} \right]
\end{aligned} \tag{3.20}$$

where

$$\gamma_0 = E_s \left(\bar{s}_{N-1}^T \Delta_{N-1}^T Q_N \Delta_{N-1} \bar{s}_{N-1} / D_{N-1} \right) \tag{3.21}$$

In the deterministic case, γ_0 is zero since Δ_n is zero for all n . This agrees with equation (3.19) for this case.

To proceed further, the information constraint must be considered. First note that in either the deterministic or random input case, the state of the

system at any sample instant is not known exactly in general. Only noisy measurements of the output at the sample instants are available in the usual case. These measurements constitute part of the data available for the estimation of the state. In particular, at time t_n , the data D_n consists of any bit of information about the state of the system, including any past record or past state.

It is known from estimation theory that the best estimate of a random variable in the sense of minimizing a quadratic error criterion is the mean of the a posteriori distribution²⁶ which in this case is the expected value of the state given the information or data D .

Using the conditional expectation then, we have

$$\bar{z}_{N-1}^* = E(\bar{z}_{N-1}/D_{N-1}) \quad (3.22)$$

and will call this estimate optimum leaving the details of the conditional expectation as an optimum estimator to Chapter 5.

Assuming that the optimal estimate of the state is available then, when applying the information constraint, equation (3.20) reduces to

$$I_1 = \underset{(\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1})}{\text{Minimum}} \left[(\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}^*)^T Q_N \dots \dots (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}^*) + \gamma_0 \right] \quad (3.23)$$

To include all cases of interest, we will let \bar{z}_{N-1}^* represent the estimate of the state in the noisy case, and the exact value of the state if it is known.

Equation (3.23) is minimized for a best pair $(\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1})$. The details of this determination is left for the general case. For the present, assume that a best or optimal pair exists. The minimum value of the quadratic form is then

$$B_1 = (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}^*)^T Q_N (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}^*) \Big|_{\substack{\text{opt} \\ (\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1})}} \quad (3.24)$$

The loss function I_1 has a value corresponding to this choice of pulse width and sign.

$$I_1 = E(B_1 + \gamma_0) \quad (3.25)$$

Continuing the step by step solution let, $n = N-1$.

$$I_2 = \underset{(\bar{p}_{N-2}, \bar{\epsilon}_{N-2})}{\text{Minimum}} \left[E(\bar{z}_{N-1}^T Q_{N-1} \bar{z}_{N-1} / D_{N-2}) + I_1 \right] \quad (3.26)$$

Using the I_1 previously determined, we obtain

$$I_2 = \underset{(\bar{p}_{N-2}, \bar{\epsilon}_{N-2})}{\text{Minimum}} \left[E(\bar{z}_{N-1}^T Q_{N-1} \bar{z}_{N-1} / D_{N-2}) + B_1 + \gamma_0 \right] \quad (3.27)$$

Equation (3.27) is analogous to equation (3.16) within the constants B_1 and γ_0 . Hence, we may follow the same procedure and include the $(B_1 + \gamma_0)$, obtaining

$$I_2 = \underset{(\bar{p}_{N-2}, \bar{\epsilon}_{N-2})}{\text{Minimum}} \left[(\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*)^T Q_{N-1} \right. \\ \left. \dots (\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*) + \gamma_1 + \gamma_0 + B_1 \right] \quad (3.28)$$

where

$$\gamma_1 = E(\bar{s}_{N-2}^T \Delta_{N-2}^T Q_{N-2} \Delta_{N-2} \bar{s}_{N-2} / D_{N-2}) \quad (3.29)$$

Defining B_2 in a manner similar to before

$$B_2 = \left. (\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*)^T Q_{N-1} (\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*) \right|_{\underset{(\bar{p}_{N-2}, \bar{\epsilon}_{N-2})}{\text{opt}}} \quad (3.30)$$

The loss function is

$$I_2 = E(B_1 + B_2 + \gamma_0 + \gamma_1) \quad (3.31)$$

The quadratic form to be minimized in either the deterministic or random case is:

$$\underset{(\bar{\rho}_{N-2}, \bar{\epsilon}_{N-2})}{\text{Minimize}} \left[(\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*)^T Q_{N-1} (\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*) \right] \quad (3.32)$$

where

$$\bar{z}_{N-2}^* = E(\bar{Z}_{N-2}/D_{N-2}) \quad (3.33)$$

Summarizing the results for the first two stages:

$n = N$

$$I_1 = E(B_1 + \gamma_0) \quad (3.34)$$

$$\underset{(\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1})}{\text{Minimize}} \left[(\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}^*)^T Q_N (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}^*) \right]$$

$$B_1 = (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}^*)^T Q_N (\bar{A}_{N-1} \bar{\epsilon}_{N-1} + \phi_{N-1} \bar{z}_{N-1}^*) \Big|_{\text{opt}(\bar{\rho}_{N-1}, \bar{\epsilon}_{N-1})}$$

$$\gamma_0 = E(\bar{s}_{N-1}^T \Delta_{N-1}^T Q_N \Delta_{N-1} \bar{s}_{N-1}/D_{N-1})$$

$n = N-1$

$$I_2 = E(B_1 + B_2 + \gamma_0 + \gamma_1) \quad (3.35)$$

$$\underset{(\bar{\rho}_{N-2}, \bar{\epsilon}_{N-2})}{\text{Minimize}} \left[(\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*)^T Q_{N-1} (\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*) \right]$$

$$B_2 = (\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*)^T Q_{N-1} (\bar{A}_{N-2} \bar{\epsilon}_{N-2} + \phi_{N-2} \bar{z}_{N-2}^*) \Big|_{\text{opt}(\bar{\rho}_{N-2}, \bar{\epsilon}_{N-2})}$$

$$\gamma_1 = E(\bar{s}_{N-2}^T \Delta_{N-2}^T Q_{N-1} \Delta_{N-2} \bar{s}_{N-2}/D_{N-2})$$

The general term for the n th point in the N stage process may simply be written now by induction.

$$I_{N-n+1} = E \sum_{i=0}^{N-n} (\gamma_i + B_{i+1}) \quad \text{for } n=1,2,\dots,N \quad (3.36)$$

The quadratic form to be minimized is

$$\begin{aligned} & \underset{(\bar{\rho}_{n-1}, \bar{\epsilon}_{n-1})}{\text{Minimize}} \left[(\bar{A}_{n-1} \bar{\epsilon}_{n-1} + \phi_{n-1} \bar{z}_{n-1}^*)^T Q_n (\bar{A}_{n-1} \bar{\epsilon}_{n-1} + \phi_{n-1} \bar{z}_{n-1}^*) \right] \\ & \text{for } n = 1, 2, \dots, N \end{aligned} \quad (3.37)$$

$$B_{N-n+1} = \left(\bar{A}_{n-1} \bar{\epsilon}_{n-1} + \phi_{n-1} \bar{z}_{n-1}^* \right)^T Q_n \left(\bar{A}_{n-1} \bar{\epsilon}_{n-1} + \phi_{n-1} \bar{z}_{n-1}^* \right) \Big|_{\text{opt}(\bar{\rho}_{n-1}, \bar{\epsilon}_{n-1})} \quad (3.38)$$

where

$$Q_n = M_n^T Q' M_n$$

and

$$\gamma_{N-n} = E (\bar{s}_{n-1}^T \Delta_{n-1}^T Q_{n-1} \Delta_{n-1} \bar{s}_{n-1} / D_{n-1}) \quad (3.39)$$

The control law will be obtained from the minimization of the quadratic form (3.37). Before accomplishing this, let us examine the significance of this particular form. Define

$$\eta_n = M_n (\bar{A}_{n-1} \bar{\epsilon}_{n-1} + \phi_{n-1} \bar{z}_{n-1}^*) \quad (3.40)$$

or

$$\eta_n = \begin{bmatrix} M_n^x & -M_n^w \end{bmatrix} \begin{bmatrix} A_{n-1} \\ 0 \end{bmatrix} \bar{\epsilon}_{n-1} + \begin{bmatrix} \phi_{n-1}^x & 0 \\ 0 & \phi_{n-1}^w \end{bmatrix} \begin{bmatrix} \bar{x}_{n-1}^* \\ \bar{w}_{n-1}^* \end{bmatrix} \quad (3.41)$$

Expanding (3.41)

$$\eta_n = M_n^x A_{n-1} \bar{\epsilon}_{n-1} + M_n^x \phi_{n-1}^x \bar{x}_{n-1}^* - M_n^w \phi_{n-1}^w \bar{w}_{n-1}^* \quad (3.42)$$

Consider the last term in (3.42). Only terms in the input process are involved. The input process was defined by (2.12).

$$\begin{aligned}\bar{w}_n &= \phi_{n-1}^w \bar{w}_{n-1} + \Delta_{n-1} \bar{s}_{n-1} \\ \bar{r}_n &= M_n^w \bar{w}_n\end{aligned}$$

Taking the conditional expectation with respect to \bar{w} we have

$$E(\bar{w}_n/D_{n-1}) = \phi_{n-1}^w E(\bar{w}_{n-1}/D_{n-1})$$

or, in simpler notation

$$\bar{w}_{n/n-1}^* = \phi_{n-1}^w \bar{w}_{n-1}^* , \quad (3.43)$$

since the process input vector satisfies the original requirements of Chapter 2. Also,

$$E(\bar{r}_n/D_{n-1}) = \bar{r}_{n/n-1}^* = M_n^w \bar{w}_{n/n-1}^* \quad (3.44)$$

Combining these terms we obtain

$$\bar{r}_{n/n-1}^* = M_n^w \bar{w}_{n/n-1}^* = M_n^w \phi_{n-1}^w \bar{w}_{n-1}^* \quad (3.45)$$

Substituting this expression into (3.42) for η_n

$$\eta_n = M_{n,n-1}^x \bar{\epsilon}_{n-1} + M_{n,n-1}^x \phi_{n-1}^x \bar{x}_{n-1}^* - \bar{r}_{n/n-1}^* \quad (3.46)$$

The remaining terms are involved with the plant process. Repeating these equations for convenience,

$$\begin{aligned}\bar{c}_n &= M_n^x \bar{x}_n \\ \bar{x}_n &= \phi_{n-1} \bar{x}_{n-1} + A_{n-1} \bar{\epsilon}_{n-1}\end{aligned} \quad (3.47)$$

Taking the conditional expectation of the first equation in (3.47), we obtain

$$E(\bar{c}_n/D_{n-1}) = \bar{c}_{n/n-1}^* = M_n^x E(\bar{x}_n/D_{n-1}) \quad (3.48)$$

or

$$\bar{c}_{n/n-1}^* = M_n^x \bar{x}_{n/n-1}^*$$

Taking the conditional expectation of the second equation,

$$\bar{x}_{n/n-1}^* = \phi_{n-1} \bar{x}_{n-1}^* + A_{n-1} \bar{e}_{n-1}$$

Thus,

$$\bar{c}_{n/n-1}^* = M_n^x \phi_{n-1} \bar{x}_{n-1}^* + M_n^x A_{n-1} \bar{e}_{n-1} \quad (3.49)$$

Using the expressions (3.45) and (3.49) in (3.42) we see that,

$$\eta_n = E(\bar{c}_n - \bar{r}_n/D_{n-1}) = \bar{c}_{n/n-1}^* - \bar{r}_{n/n-1}^*$$

or

$$\eta_n = \bar{e}_{n/n-1}^* = E(\bar{e}_n/D_{n-1}) \quad (3.50)$$

The function η_n is simply the expected value of the error at time t_n given the data at time t_{n-1} .

The quadratic form to be minimized then simply represents the measure of system error at one sample instant ahead of the time at which the pulse will be applied. Obviously, the dynamic programming approach here simply provides the organized logical structure to follow in arriving at the final result.

3.5 The Control Law

The minimization of the quadratic form (3.37) results in the control law for optimal behavior of the system. Summarizing the pertinent equations,

$$\underset{(\bar{\phi}_{n-1}, \bar{e}_{n-1})}{\text{Minimize}} \left[\eta_n^T Q_n' \eta_n \right] \quad (3.51)$$

where

$$\eta_n = M_n^x A_{n-1} \bar{e}_{n-1} + M_n^x \phi_{n-1}^x \bar{x}_{n-1}^* - \bar{r}_{n/n-1}^* \quad (3.52)$$

Let the function H_n^* be defined as

$$H_n^* = \bar{r}_{n/n-1}^* - M_n^x \phi_{n-1}^x \bar{x}_{n-1}^* \quad (3.53)$$

This function then takes into account the results of the optimum estimation of the state of the system given the data D_{n-1} . The calculations and details of obtaining H_n^* are accomplished in Chapter 4. We will assume temporarily that the results of the estimation are available and the function H_n^* is known. The expression for η_n using the results of the estimator is thus

$$\eta_n = M_n^x A_{n-1} \bar{\epsilon}_{n-1} - H_n^* \quad (3.54)$$

Using the expression for A_{n-1} we obtain

$$\eta_n = M_n^x \phi_{n-1}^x \int_0^{\bar{\rho}_{n-1})_i} e^{-A\lambda} B \, d\lambda \bar{\epsilon}_{n-1} - H_n^* \quad (3.55)$$

for $i = 1, 2, \dots, q$

Define

$$F(\bar{\rho}_{n-1}, t_n - t_{n-1}) = M_n^x \phi_{n-1}^x \int_0^{\bar{\rho}_{n-1})_i} e^{-A\lambda} B \, d\lambda \quad (3.56)$$

for $i = 1, 2, \dots, q$

Equation (3.54) may now be written

$$\eta_n = F(\bar{\rho}_{n-1}, t_n - t_{n-1}) \bar{\epsilon}_{n-1} - H_n^*$$

To simplify the notation, let

$$\sigma_n = \text{the length of the } n\text{th sampling interval}$$

or

$$\sigma_n = t_{n+1} - t_n$$

Then,

$$\phi_{n-1}^x = \phi^x(t_n - t_{n-1}) = \phi^x(\sigma_n)$$

and since the output matrix does not depend on n

$$M_n^x = M^x$$

Equation (3.56) reduces to

$$\eta_n = F(\bar{\rho}_{n-1}, \sigma_n) \bar{\epsilon}_{n-1} - H_n^* \quad (3.57)$$

Instead of referring to the n th sampling interval, the notation may be further simplified by referring to any sampling interval, and we may write

$$\eta = F(\bar{\rho}, \sigma) \bar{\epsilon} - H^* \quad (3.58)$$

where

- σ = the length of the sampling interval
- $\bar{\rho}$ = the pulse width vector associated with the particular sampling interval
- $\bar{\epsilon}$ = the sign vector associated with the particular sampling interval.

Returning to the quadratic form with this new notation, we have

$$\text{Minimize}_{(\bar{\rho}, \bar{\epsilon})} [\eta^T Q' \eta]$$

or

$$\text{Minimize}_{(\bar{\rho}, \bar{\epsilon})} [F(\bar{\rho}, \sigma) - H^*]^T Q' [F(\bar{\rho}, \sigma) - H^*] \quad (3.59)$$

The condition to be taken to insure minimization of (3.59) will be the control law. The control law then must satisfy

$$\text{Minimize}_{(\bar{\rho}, \bar{\epsilon})} [F(\bar{\rho}, \sigma) \bar{\epsilon} - H^*] \quad (3.60)$$

The function H^* is known from the results of estimation of the states of both the plant and input processes. Knowing the length of the sampling interval or using the best estimate available, we have sufficient information to satisfy condition (3.60) by choice of the pulse width and sign which will be optimal in the sense defined here.

It simply remains to investigate the function $F(\bar{\rho}, \sigma)$ for various types of systems since the estimation problem is covered in Chapter 4. The sign vector $\bar{\epsilon}$ may easily be determined for the situation at hand and this bears no further comment.

For a single variable control, the function $F(\rho, \sigma)$ takes a particularly simple form

$$F(\rho, \sigma) = M^x \phi^x(\sigma) \int_0^{\rho} e^{-A p} B \, dp \quad (3.61)$$

If the output of the system is directly related to one of the state variables the output matrix is quite simple and fortunately, this is the usual case. In particular,

$$M^x = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and since the order of the numerator is usually less than that of the denominator for the plant transfer function, the matrix B also contains several zeros. The calculation of $F(\rho, \sigma)$ is then quite simple and only a little more involved than the calculation of the characteristic matrix which may be accomplished easily by means of Mason's signal flow graph.³⁴

The function $F(\rho, \sigma)$ is shown in Table 1 for several typical plant processes. Since $\rho \leq \sigma$ at all times, the boundary curve may be considered to divide the mode of operation from on-off control to pulse width control. It is apparent that in some cases, the expression for pulse width may be obtained in closed form as a function of the estimation H^* and the sampling interval σ . This expression will satisfy the condition for optimality and constitutes the control law in these cases.

TABLE 1

| $G(s)$ | $F(\rho, \sigma) \quad \rho \leq \sigma$ | Boundary $\rho = \sigma$ |
|-------------------------|--|---|
| $\frac{1}{s+1}$ | $e^{-\sigma} (e^{\rho} - 1)$ | $1 - e^{-\rho}$ |
| $\frac{1}{s(s+1)}$ | $\rho - e^{-\sigma} (e^{\rho} - 1)$ | $\rho - (1 + e^{-\rho})$ |
| $\frac{1}{s^2(s+1)}$ | $\rho(\sigma+1) - \frac{\rho^2}{2} + e^{-\sigma}(e^{\rho} + 1)$ | $\rho(\frac{\rho}{2} + 1) + 1 + e^{-\rho}$ |
| $\frac{1}{s(s+1)(s+2)}$ | $\frac{\rho}{2} + e^{-\sigma}(1 - e^{\rho}) + \frac{e^{-2\sigma}}{2}(1 - e^{2\rho})$ | $\frac{\rho}{2} + \frac{(e^{-\rho} - 3)(e^{-\rho} + 1)}{4}$ |

The technique which results in optimal control can be reviewed easily using the curves showing the relationship between pulse width, sampling length and the function $F(\rho, \sigma)$. Following the idea of using the pulse width as a parameter, the curves may be presented as shown in Figure 2.

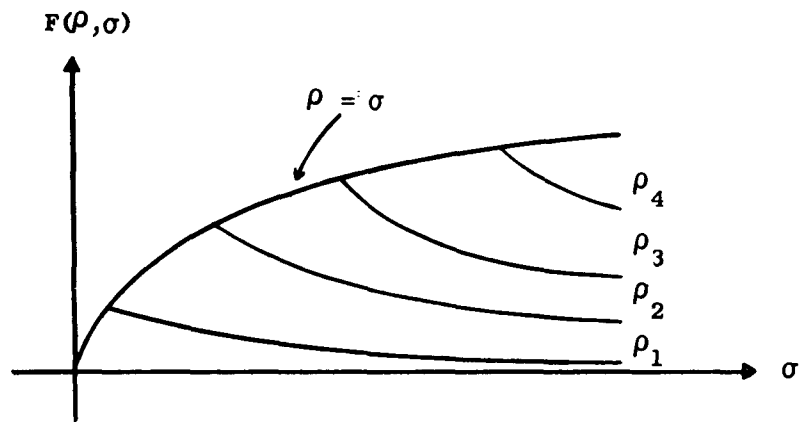


Figure 2. Typical Characteristic Curve

For a known value of the estimation function H^* and the sampling length, the optimal pulse width and sign are determined so that the condition for optimality is satisfied. Repeating this for convenience we have

$$\text{Minimize}_{(\rho, \epsilon)} [F(\rho, \sigma)\epsilon - H^*] \quad (3.60)$$

These "characteristic" curves are shown in Figures 3 to 6 for several systems including ones to be used as examples in Chapter 6.

3.6 Controller Mechanization

The program or mechanization of the procedure reviewed in terms of the "characteristic" curves constitutes the design of the optimal controller. The control computer may be programmed to execute a sequence of logical orders for each sampling instant, determining the optimal pair (ρ, ϵ) . A flow graph of a general scheme is shown in Figure 7 where the inputs are the length of the sampling interval or its best estimate, and the optimal estimate of the state \bar{z}^* .

Let us consider possible realizations for two simple classes of systems with emphasis on the calculations to be executed for each sampling instant.

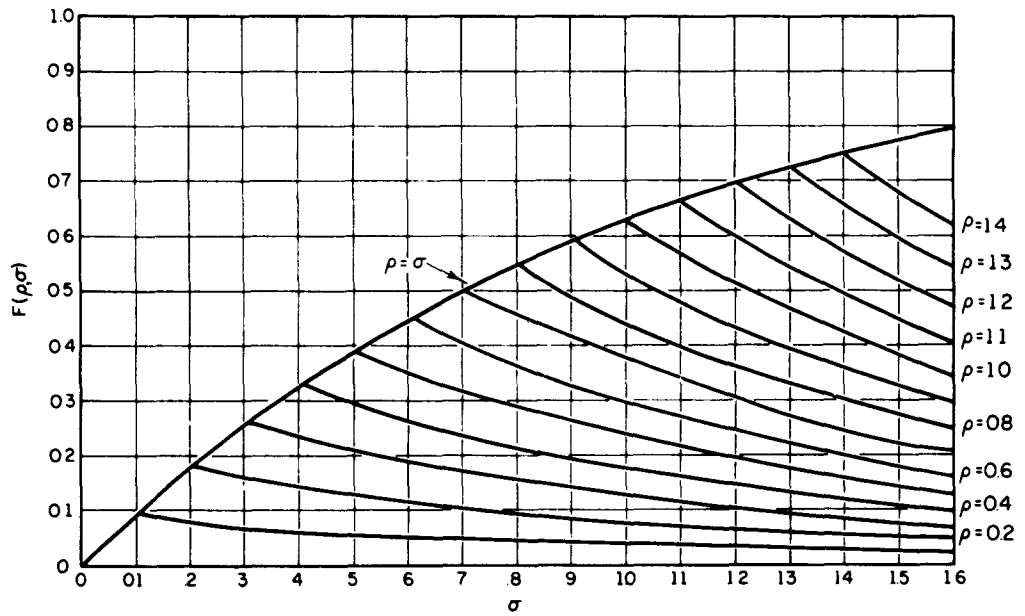


Figure 3. Characteristic Curves for $G(s) = \frac{1}{s+1}$

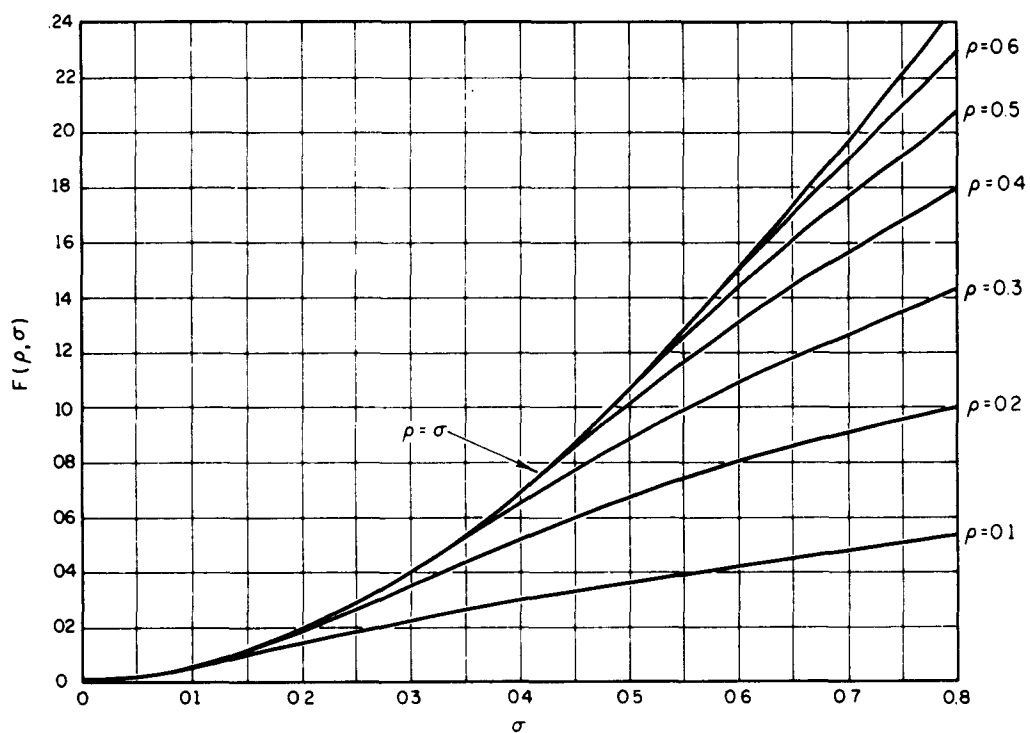


Figure 4. Characteristic Curves for $G(s) = \frac{1}{s(s+1)}$

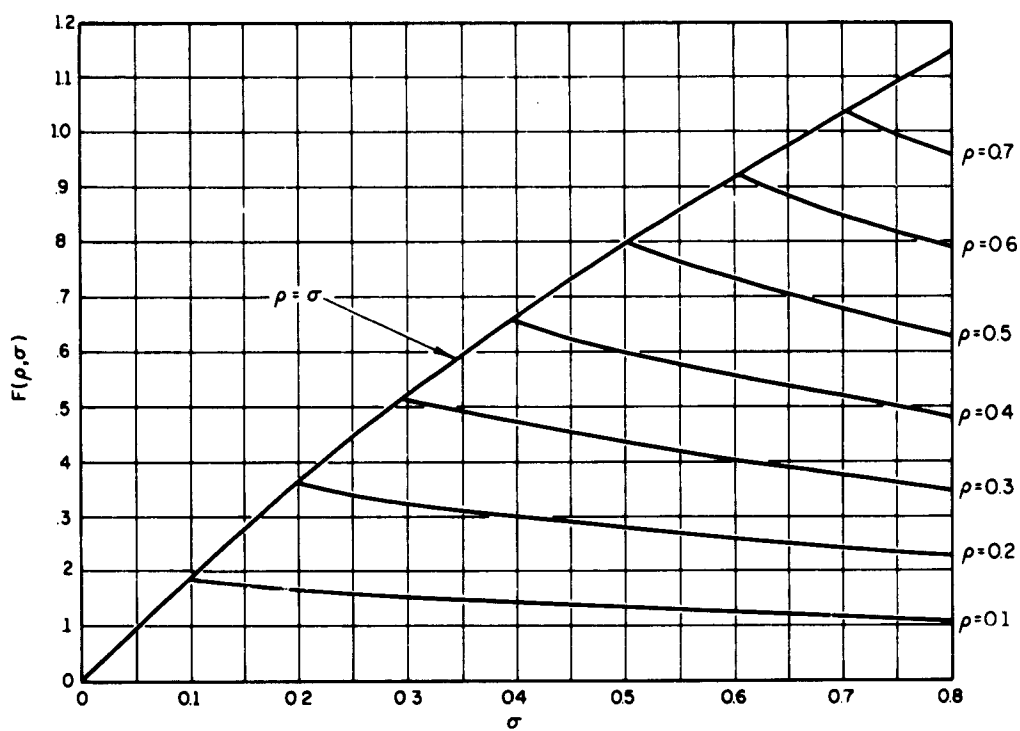


Figure 5. Characteristic Curves for $G(s) = \frac{1}{s(s+1)(s+2)}$

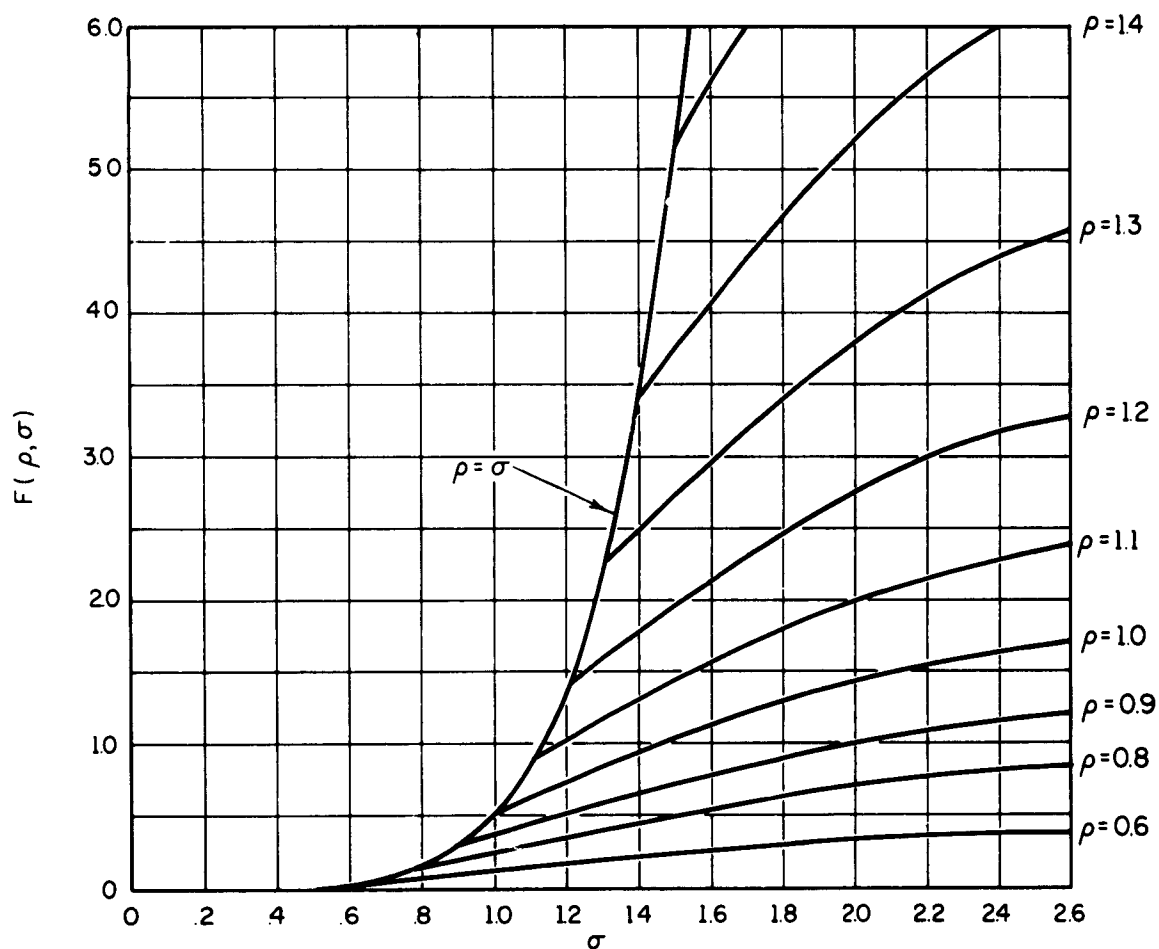


Figure 6. Characteristic Curves for $G(s) = \frac{1}{s(s+3)(s^2 + 2s + 2)}$

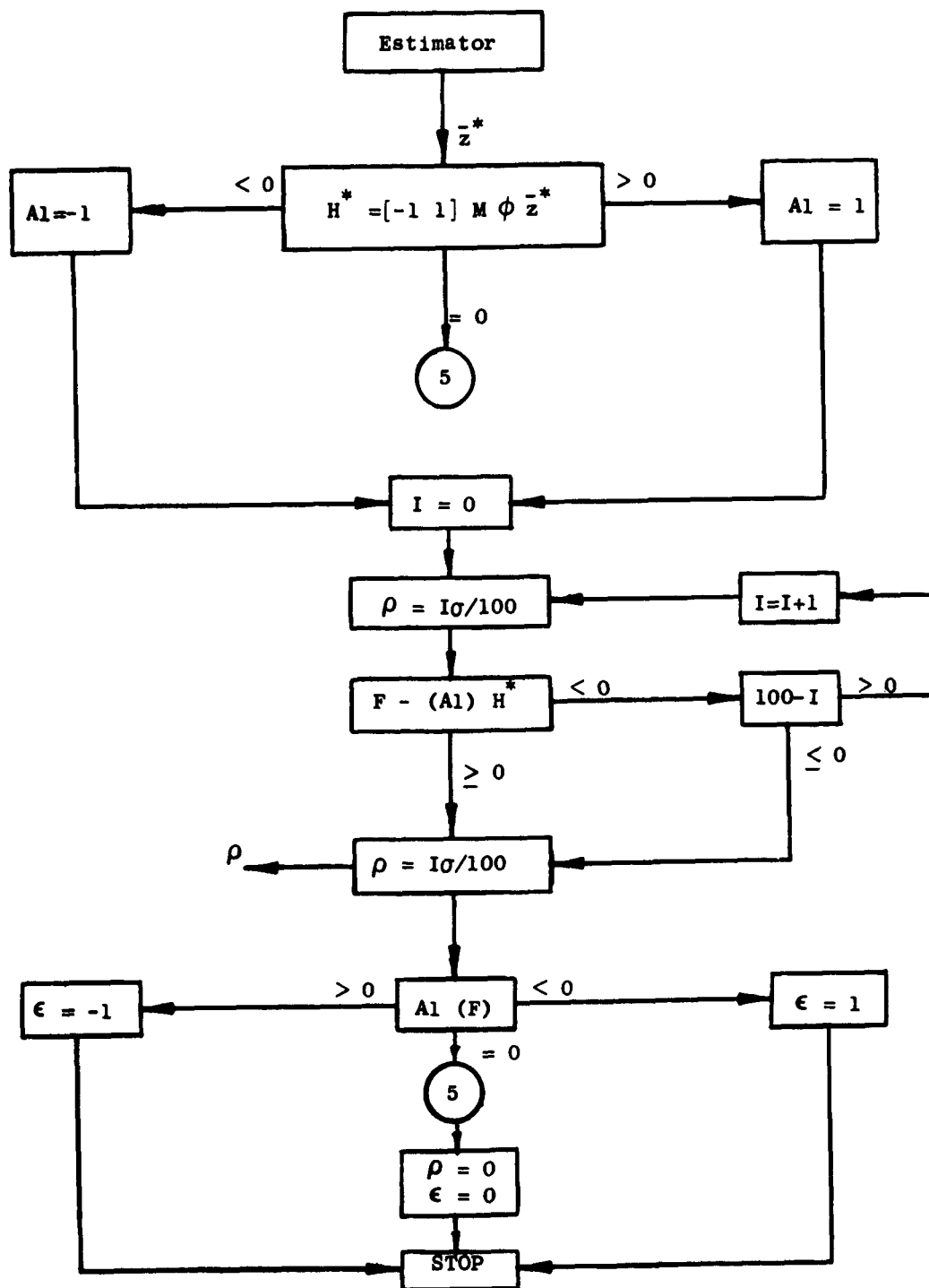


Figure 7. Flow Graph of Controller Where $F = F(\rho, \sigma)$

First consider a type zero system with a transfer function

$$G(s) = \frac{K}{s + a}$$

The transition, distribution and output matrices are:

$$A = -a \quad B = K \quad M = 1$$

The function $F(\rho, \sigma)$ is

$$F(\rho, \sigma) = M \phi(\sigma) \int_0^\rho e^{-Ap} B dp = e^{-a\sigma} \frac{K}{a} (e^{a\rho} - 1) \quad (3.62)$$

In this case $F(\rho, \sigma) \geq 0$ for $0 \leq \rho \leq \sigma$

Thus,

$$\epsilon = \text{Sign of } (H^*) = \text{sgn}(H^*) \quad (3.63)$$

To achieve a zero minimum of equation (3.60), the pulse width is

$$\rho = \frac{1}{a} \text{Ln} \left[1 + \frac{a}{K} e^{a\sigma} H^* \epsilon \right] \quad (3.64)$$

Writing H^* in terms of the optimal estimate \bar{z}^* we have,

$$H^* = \begin{bmatrix} -1 & 1 \end{bmatrix} M \phi(\sigma) \bar{z}^* \quad (3.65)$$

The logarithm in equation (3.64) must be calculated to determine a pulse width, and then checked to insure the condition

$$0 \leq \rho \leq \sigma$$

The complete realization for the controller is shown in Figure 8. In this realization, the voltages representing the pulse width ρ and sign ϵ are multiplied together and applied to a hold circuit providing a staircase output which is the desired input signal to the pulse width modulator. These waveforms are shown in Figure 9. The complete block diagram of the system is shown in Figure 10. If the state is known exactly, no filter is required.

In this simple case, the value of ρ has been obtained in closed form. This will not be possible in general.

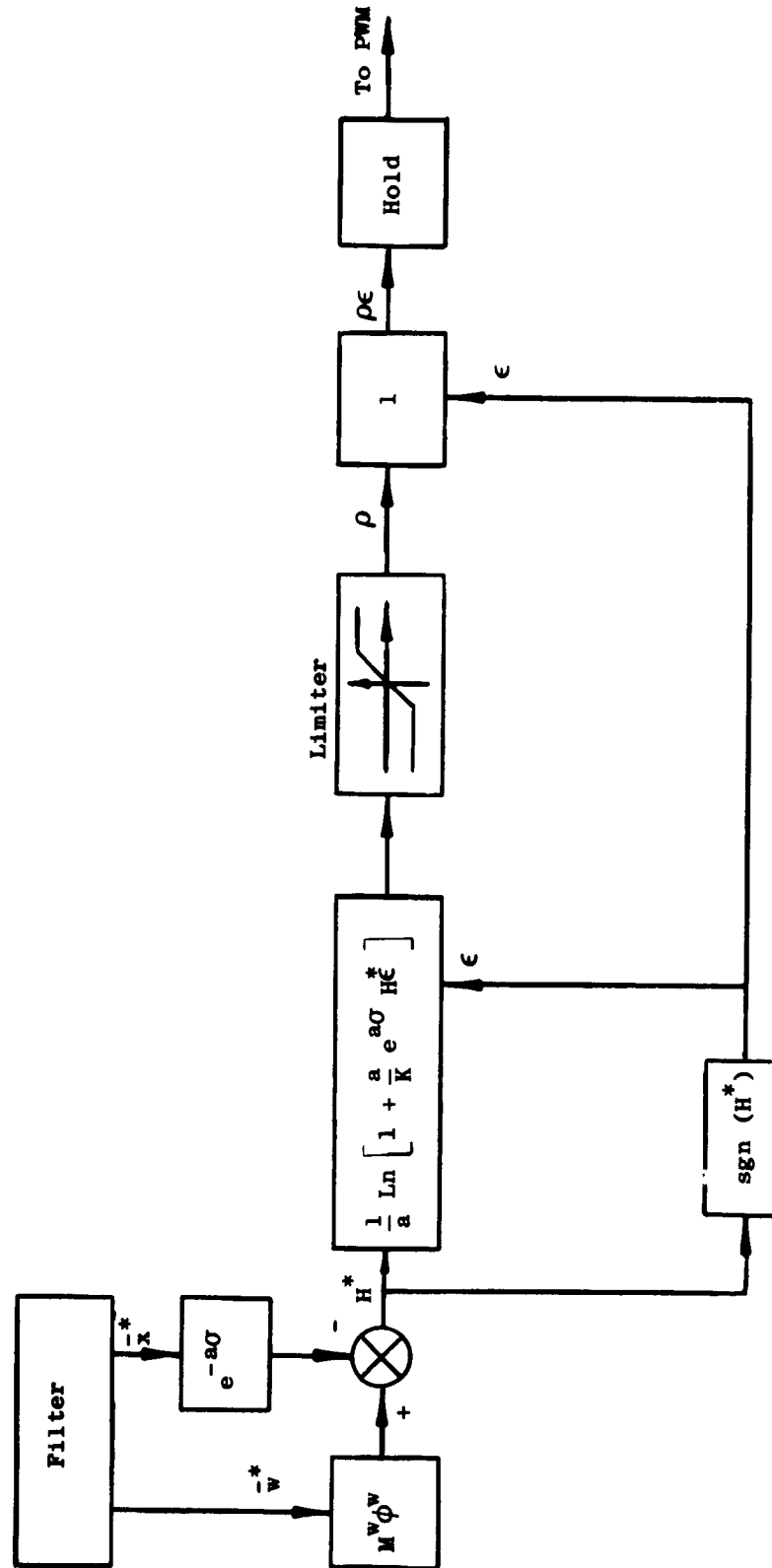


Figure 8. Controller Realization for $G(s) = \frac{K}{s+a}$

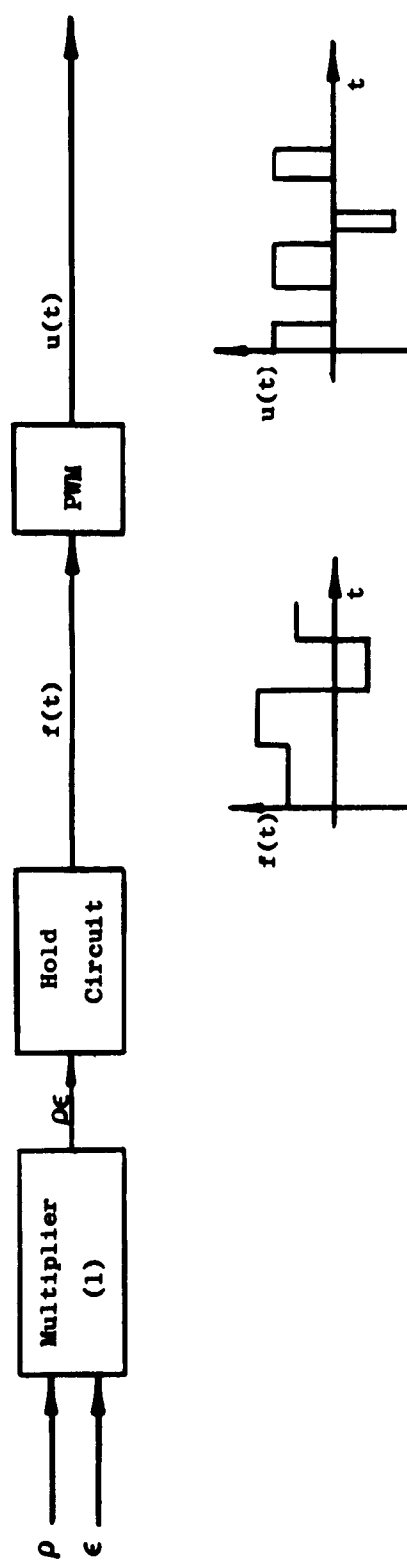


Figure 9. Waveforms of Controller and PWM Output

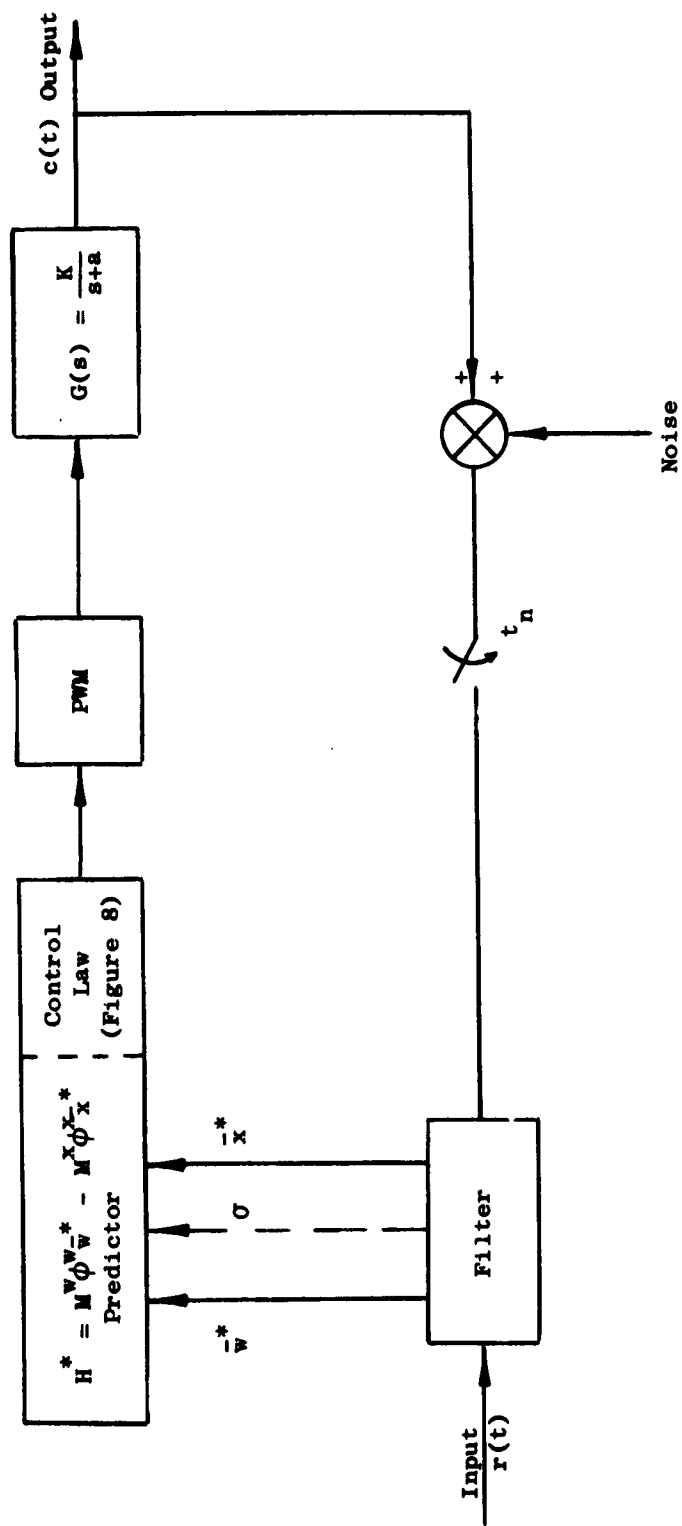


Figure 10. Block Diagram of Complete System with a First Order Plant

Consider a type one second order class of systems with the transfer function

$$G(s) = \frac{K}{s(s+a)}$$

The transition, distribution and output matrices are:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ K \end{bmatrix} \quad M = [1 \quad 0]$$

The characteristic matrix is

$$\phi(t) = e^{At} = \begin{bmatrix} 1 & \frac{1}{a} - \frac{e^{-at}}{a} \\ 0 & e^{-at} \end{bmatrix}$$

The function $F(\rho, \sigma)$ is

$$F(\rho, \sigma) = M\phi(\sigma) \int_0^\rho e^{-Ap} B dp = \frac{K}{a^2} [a\rho - e^{-a\sigma}(e^{a\rho} - 1)] \quad (3.66)$$

where

$$0 \leq \rho \leq \sigma$$

The control law is determined from equation (3.60)

$$\underset{(\rho, \sigma)}{\text{Minimize}} [F(\rho, \sigma)\epsilon - H^*] \quad (3.60)$$

For a zero minimum,

$$\frac{K}{a^2} [a\rho - e^{-a\sigma}(e^{a\rho} - 1)] = H^*\epsilon \quad (3.67)$$

or

$$(a\rho) - (e^{-a\sigma}) e^{(a\rho)} = \frac{a^2}{K} H^*\epsilon - (e^{-a\sigma}) \quad (3.68)$$

Since $F(\rho, \sigma) \geq 0$ for $0 \leq \rho \leq \sigma$, $a > 0$ and $K > 0$, then

$$\epsilon = \text{Sign of } (H^*) = \text{sgn}(H^*) \quad (3.69)$$

Equation (3.60) may be minimized by a program shown in Figure 11. The variable is incremented and the function tested at each step. The program is repeated for each sampling interval.

Another possible realization may be accomplished with analog elements. The expression

$$f(a\rho) = (a\rho) - (e^{-a\sigma}) e^{(a\rho)}$$

is the solution to a constant coefficient ordinary differential equation which is easily programmed. The value of the function is compared to the remaining terms in (3.68) and if a zero minimum condition is satisfied, the computer is stopped with $(a\rho)$ held on the output of an integrator. Provision is also included for minimization on the boundary where $\rho = \sigma$. The realization is particularly simple since $f(a\rho)$ is well behaved with

$$\begin{aligned} \dot{f}(a\rho) &> 0 \\ \dot{f}(a\rho) &< 0 & \text{for } 0 \leq \rho < \sigma \\ \dot{f}(a\rho) &= 0 & \text{for } \rho = \sigma \end{aligned}$$

eliminating the possibility of a minimum not on the boundary for $\rho = \sigma$. This realization is shown in Figure 12. Time scaling may be included to provide the desired response time.

If reset does not occur until new data is available, then the holding property of the integrators is sufficient to eliminate the need for a separate hold circuit. Hence, a staircase wave is fed directly to the pulse width modulator. The block diagram for the complete system is shown in Figure 13.

If exact measurement of the state occurs and the input to the system is deterministic, then no filter is needed. This is shown in Figure 14.

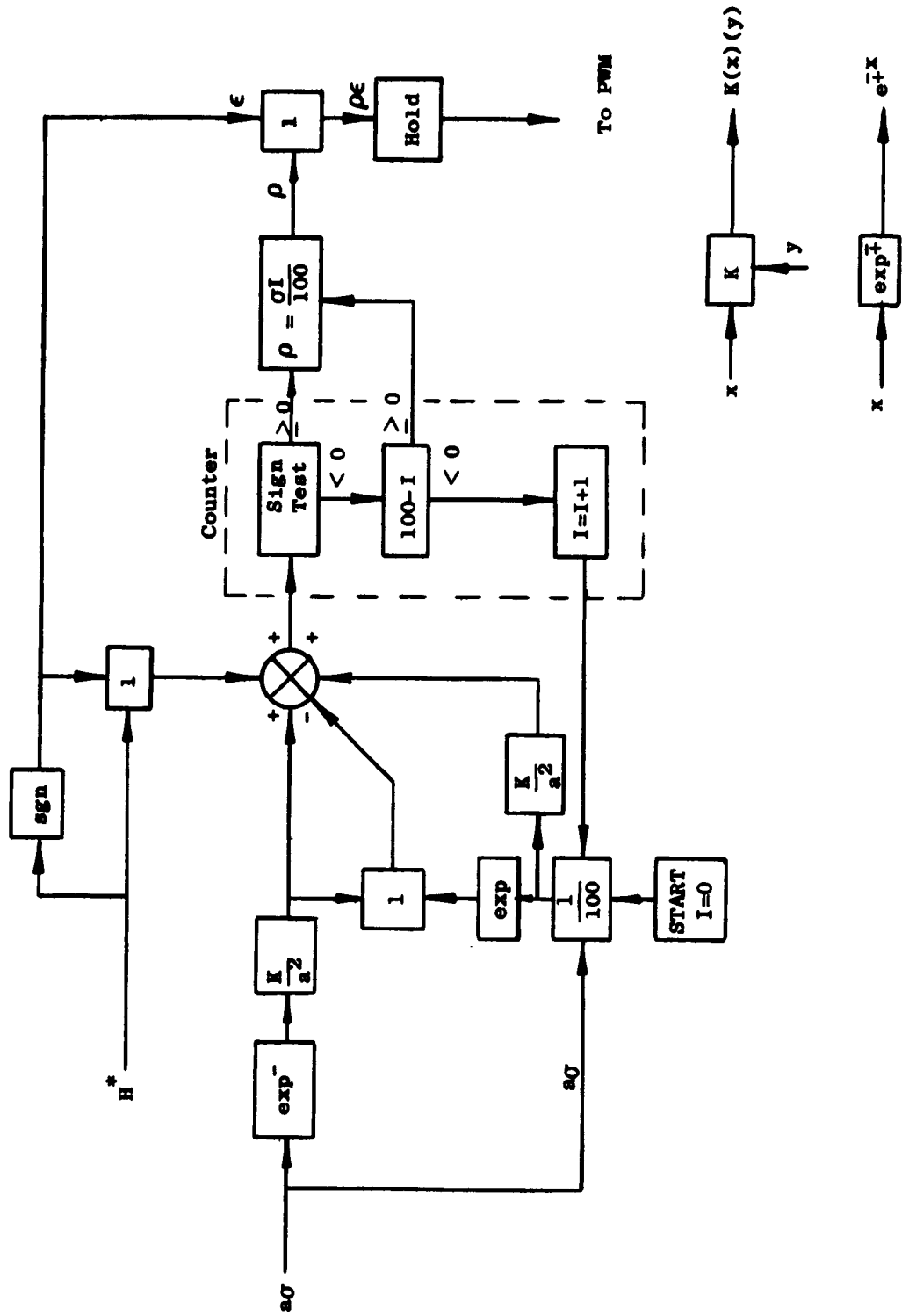


Figure 11. Controller Realization for $G(s) = \frac{K}{s(s+a)}$

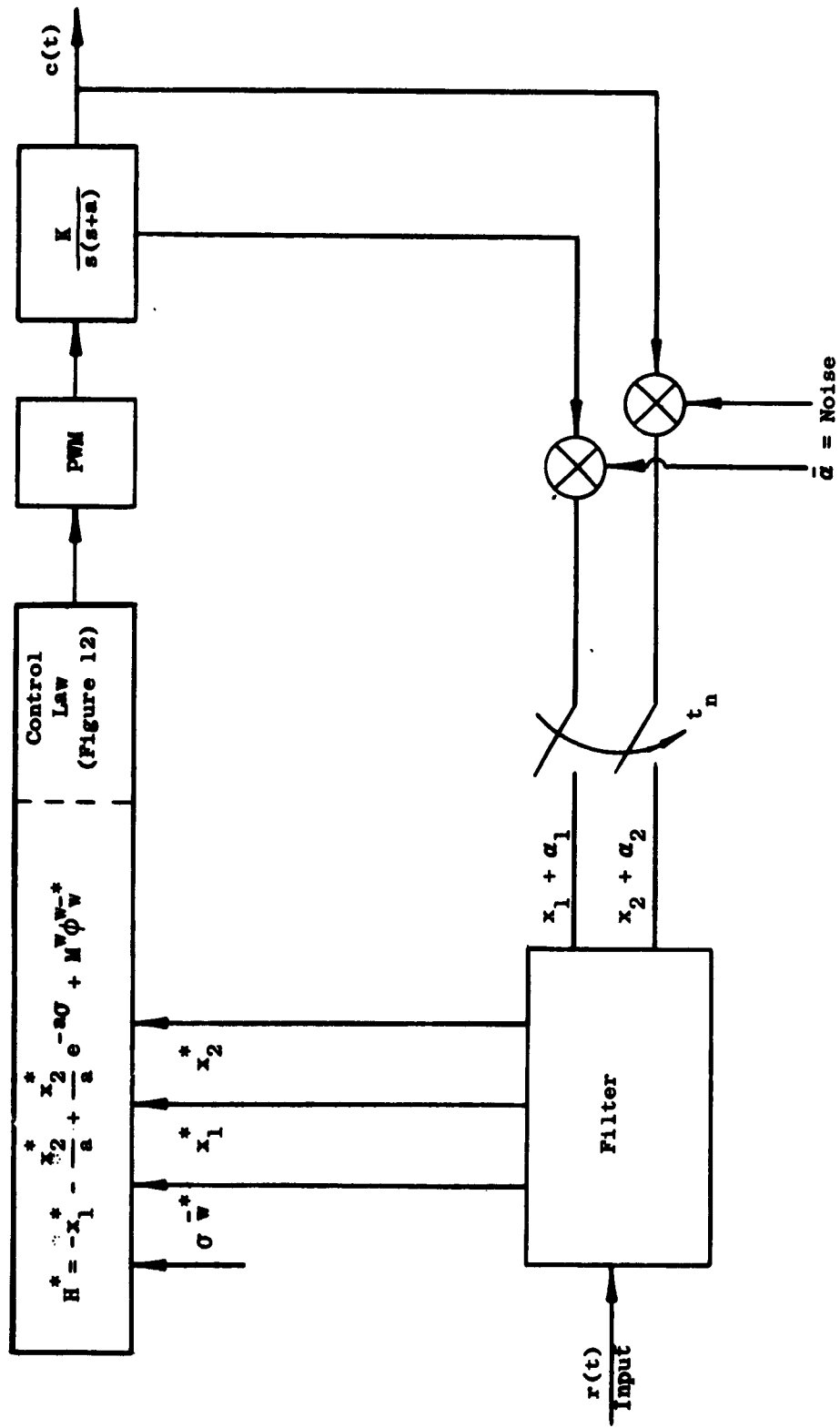


Figure 13. Block Diagram for the Complete System with a Second Order Plant

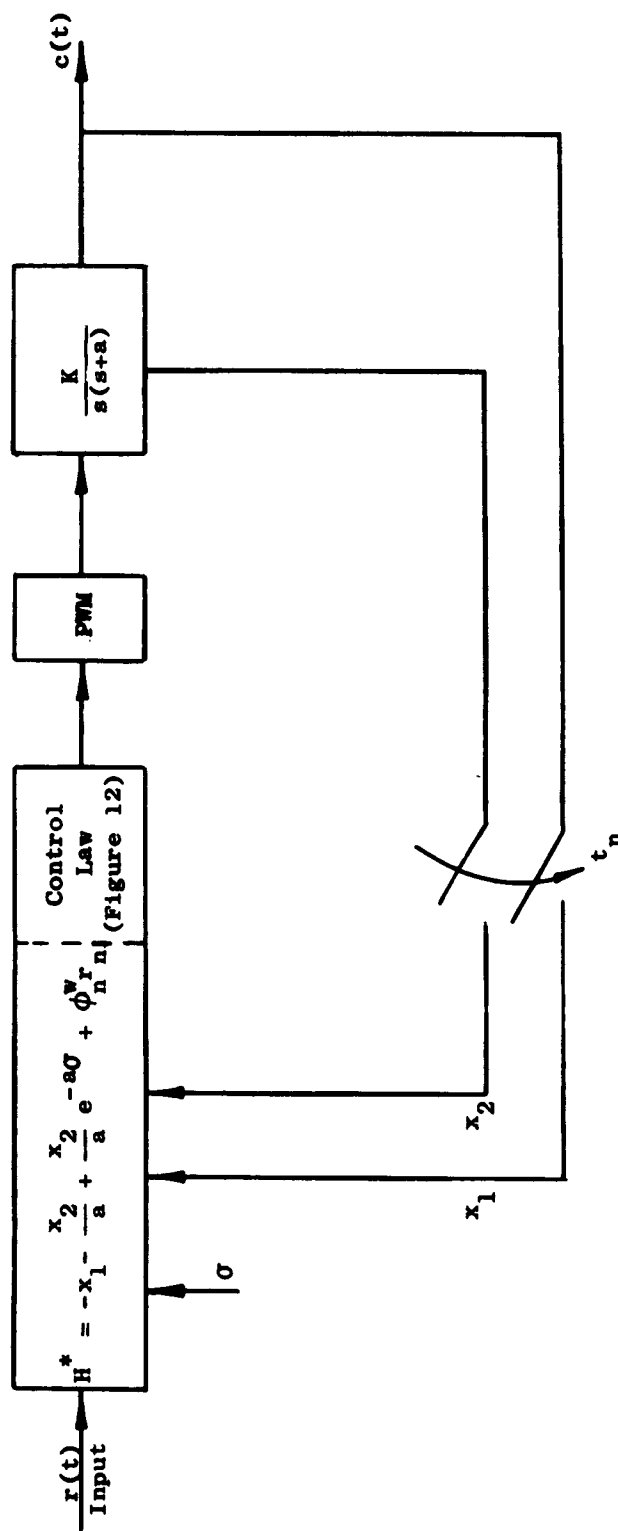


Figure 14. Block Diagram for the Complete System with No Additive Noise and Only Deterministic Inputs

4. OPTIMUM PLANT IDENTIFICATION

4.1 General

The plant requires more state variables in its description than there are measurable quantities. Since the plant state variables must be known for complete knowledge of the plant's behavior, the problem of identifying the state from the measurable outputs is apparent. The identification problem is equivalent to finding an optimum estimator for the state vector \bar{z}_k given knowledge of the output vector \bar{y}_k' for $t \leq t_k$. The initial derivation of the optimum linear estimator is due to Kalman^{26,27} and the presentation here is simply a review of the derivation with emphasis on the interpretation of results applicable to this problem.

4.2 The Conditional Mean as an Optimum Estimator

The conditional expectation $E(\bar{z}_k/\bar{y}_k', \bar{y}_{k-1}', \dots, \bar{y}_0')$ has the following properties:

- (1) It is a linear function of the output vector, requiring knowledge of only means and covariances.
- (2) If \bar{z} and \bar{y} are gaussian random vectors, it is an optimum estimator for any reasonable performance index.
- (3) If \bar{z} and \bar{y}' are not gaussian, it is an optimum linear estimator.

For these reasons, the conditional expectation will be used as the optimum linear estimator to accomplish the identification of the state of the system.

Notational Definitions:

$$\begin{aligned}\bar{z}_n^* &= E(\bar{z}_n/\bar{y}_n', \bar{y}_{n-1}', \dots, \bar{y}_0') \\ \bar{z}_{n/n-1}^* &= E(\bar{z}_n/\bar{y}_{n-1}', \bar{y}_{n-2}', \dots, \bar{y}_0') \\ \bar{e}_n &= \bar{z}_{n/n-1}^* - \bar{z}_n = \text{predicted estimation error} \\ \bar{e}_n^* &= \bar{z}_n^* - \bar{z}_n = \text{estimation error} \\ \text{cov}(\bar{z}_n, \bar{y}_n) &= E[(\bar{z}_n - E(\bar{z}_n))(\bar{y}_n - E(\bar{y}_n))^T] \\ P_n^* &= \text{cov}(\bar{e}_n^*, \bar{e}_n^*) = \text{the error covariance matrix}\end{aligned}$$

$$P_{n/n-1} = \text{cov} (\bar{e}_n, \bar{e}_n)$$

$$D_{n-1} = \{\bar{y}'_{n-1}, \bar{y}'_{n-2}, \dots, \bar{y}'_0\} = \text{set of measurements at } t = t_{n-1}$$

To show that the conditional expectation is an optimal linear estimator, consider a simple case. Define the loss function L , where Q is a positive, semi-definite, symmetric matrix.

$$L = E \left[(\bar{z} - \bar{z}^*)^T Q (\bar{z} - \bar{z}^*) / \bar{y}' \right] \quad (4.1)$$

where \bar{z}^* is an estimate of \bar{z} .

By simple manipulation, L may be written

$$L = [\bar{z}^* - E(\bar{z}/\bar{y}')]^T Q [\bar{z}^* - E(\bar{z}/\bar{y}')] - E(\bar{z}/\bar{y}')^T Q E(\bar{z}/\bar{y}') + E(\bar{z}^T Q \bar{z} / \bar{y}')$$

The dependence of the estimate \bar{z}^* is isolated in this form. Thus, the one value of \bar{z}^* which minimizes L is clearly

$$\bar{z}^* = E(\bar{z}/\bar{y}') \quad (4.2)$$

The estimate is optimum in the sense of minimizing L and is unique if Q is nonsingular. Thus, we speak of the optimum linear estimate \bar{z}^* .

4.3 The Optimum Estimator, Predictor and Filter

We will consider first the problem of estimating the state of the system from noisy measurements of the output assuming certain a priori information. The optimum estimate or mean of the a posteriori distribution is calculated in terms of the assumed a priori information. The second step is then to show how the a priori information may be developed recursively.

Consider the output relationship

$$\bar{y}_n = M_n \bar{z}_n \quad (4.3)$$

where

$$\bar{y}'_n = M_n \bar{z}_n + \bar{a}_n \quad (4.4)$$

and \bar{y}'_n is the noisy measurement at $t = t_n$ available for estimation purposes.

Since the conditional expectation is linear with respect to the output vector, the form assumed for the optimal estimate is

$$\bar{z}_n^* = E(\bar{z}_n/D_n) = \bar{z}_{n/n-1}^* + b_n (\bar{y}_n' - \bar{y}_{n/n-1}^*) \quad (4.5)$$

where b_n is determined by minimizing the covariance of the estimation error. The a priori information assumed is

$$\begin{aligned} & \bar{z}_{n/n-1}^*, E(\bar{\alpha}_n), \text{cov}(\bar{\alpha}_n, \bar{\alpha}_n) \text{ and} \\ & P_{n/n-1}^* = \text{cov}(\bar{e}_n, \bar{e}_n) = \text{covariance of predicted estimate} \end{aligned}$$

and

$$\begin{aligned} b_n & \text{ is a } (m \times p) \text{ matrix} \\ I & \text{ is an } (m \times m) \text{ unit matrix} \\ P_n & \text{ is a } (p \times p) \text{ matrix} \end{aligned}$$

The covariance of estimation error is to be minimized with respect to the elements of the b_n matrix. The error in estimation is

$$\bar{e}_n^* = \bar{z}_n^* - \bar{z}_n \quad (4.6)$$

Substituting (4.4) and (4.5) into (4.6),

$$\bar{e}_n^* = \bar{z}_{n/n-1}^* - \bar{z}_n + b_n (M_n \bar{z}_n + \bar{\alpha}_n - \bar{y}_{n/n-1}^*) \quad (4.7)$$

Since the noise is additive, we may write

$$\bar{e}_n^* = (\bar{z}_{n/n-1}^* - \bar{z}_n) + b_n (M_n \bar{z}_n + \bar{\alpha}_n) - b_n (M_n \bar{z}_{n/n-1}^* + E(\bar{\alpha}_n))$$

or

$$\bar{e}_n^* = (I - b_n M_n) (\bar{z}_{n/n-1}^* - \bar{z}_n) + b_n (\bar{\alpha}_n - E(\bar{\alpha}_n)) \quad (4.8)$$

Computing the covariance of the error in estimation

$$P_n^* = \text{cov}(\bar{e}_n^*, \bar{e}_n^*) = (I - b_n M_n) P_{n/n-1} (I - b_n M_n)^T + b_n \text{cov}(\bar{\alpha}_n, \bar{\alpha}_n) b_n^T \quad (4.9)$$

The covariance matrix P_n^* is symmetric and thus, minimization of the trace with respect to the elements of b_n is sufficient for minimizing the matrix. A necessary condition for minimizing P_n^* is found to be

$$b_n = P_{n/n-1} M_n^T (M_n P_{n/n-1} M_n^T + \text{cov}(\bar{a}_n, \bar{a}_n))^{-1}$$

Defining

$$F_n = M_n P_{n/n-1} M_n^T + \text{cov}(\bar{a}_n, \bar{a}_n) \quad (4.10)$$

the condition may be written

$$b_n = P_{n/n-1} M_n^T F_n^{-1} \quad (4.11)$$

The resulting minimum value for P_n^* when using the value of b_n from (4.10) is

$$P_n^* = P_{n/n-1} - b_n F_n b_n^T \quad (4.12)$$

The value of P_n^* is the minimum covariance of estimation error attained by the optimum linear estimator.

Summarizing the filter equations:

$$\bar{z}_n^* = (I - b_n M_n) \bar{z}_{n/n-1}^* + b_n \bar{y}_n' - b_n E(\bar{a}_n) \quad (4.13)$$

where

$$F_n = M_n P_{n/n-1} M_n^T + \text{cov}(\bar{a}_n, \bar{a}_n)$$

$$b_n = P_{n/n-1} M_n^T F_n^{-1}$$

and the minimum error covariance for this estimate is

$$P_n^* = P_{n/n-1} - b_n F_n b_n^T$$

The filter equations in (4.13) depend on the assumed a priori information $\bar{z}_{n/n-1}^*$ and $P_{n/n-1}$. This information will be developed recursively. Consider the process

$$\bar{z}_n = \phi_{n-1} \bar{z}_{n-1} + \bar{q}_{n-1} \quad (4.14)$$

Assuming P_{n-1}^* and z_{n-1}^* are known, $P_{n/n-1}$ and $z_{n/n-1}^*$ must be calculated. Taking the conditional expectation of (4.14)

$$E(\bar{z}_n / D_{n-1}) = \phi_{n-1} E(\bar{z}_{n-1} / D_{n-1}) + E(\bar{q}_{n-1} / D_{n-1})$$

or

$$\bar{z}_{n/n-1}^* = \phi_{n-1} \bar{z}_{n-1}^* + \bar{q}_{n-1}^* \quad (4.15)$$

Calculating the errors in this estimate, we have

$$\bar{e}_n = (\bar{z}_{n/n-1}^* - \bar{z}_n) = \phi_{n-1} \bar{e}_{n-1}^* + (\bar{q}_{n-1}^* - \bar{q}_{n-1}) \quad (4.16)$$

The covariance in this estimate is

$$P_{n/n-1} = \text{cov}(\bar{e}_n, \bar{e}_n) = \phi_{n-1} P_{n-1}^* \phi_{n-1}^T + E[(\bar{q}_{n-1}^* - \bar{q}_{n-1})(\bar{q}_{n-1}^* - \bar{q}_{n-1})^T] \quad (4.17)$$

Thus, the a priori information is determined from (4.15) and (4.17).

The complete estimator may now be summarized for the process

$$\bar{z}_n = \phi_{n-1} \bar{z}_{n-1} + \bar{q}_{n-1}$$

$$\bar{y}'_n = M_n \bar{z}_n + \bar{\alpha}_n$$

Given the estimate of the state at $t = t_{n-1}$ and the covariance of this estimate, \bar{z}_{n-1}^* and P_{n-1}^*

$$\bar{z}_{n/n-1}^* = \phi_{n-1} \bar{z}_{n-1}^* + \bar{q}_{n-1}^* \quad (4.18)$$

and

$$P_{n/n-1} = \phi_{n-1} P_{n-1}^* \phi_{n-1}^T + \text{cov}(\bar{q}_{n-1}, \bar{q}_{n-1})$$

The filter is

$$\bar{z}_n^* = (I - b_n M_n) \bar{z}_{n/n-1}^* - b_n \bar{y}'_n - b_n E(\bar{\alpha}_n)$$

where

$$F_n = M_n P_{n/n-1} M_n^T + \text{cov}(\bar{\alpha}_n, \bar{\alpha}_n)$$

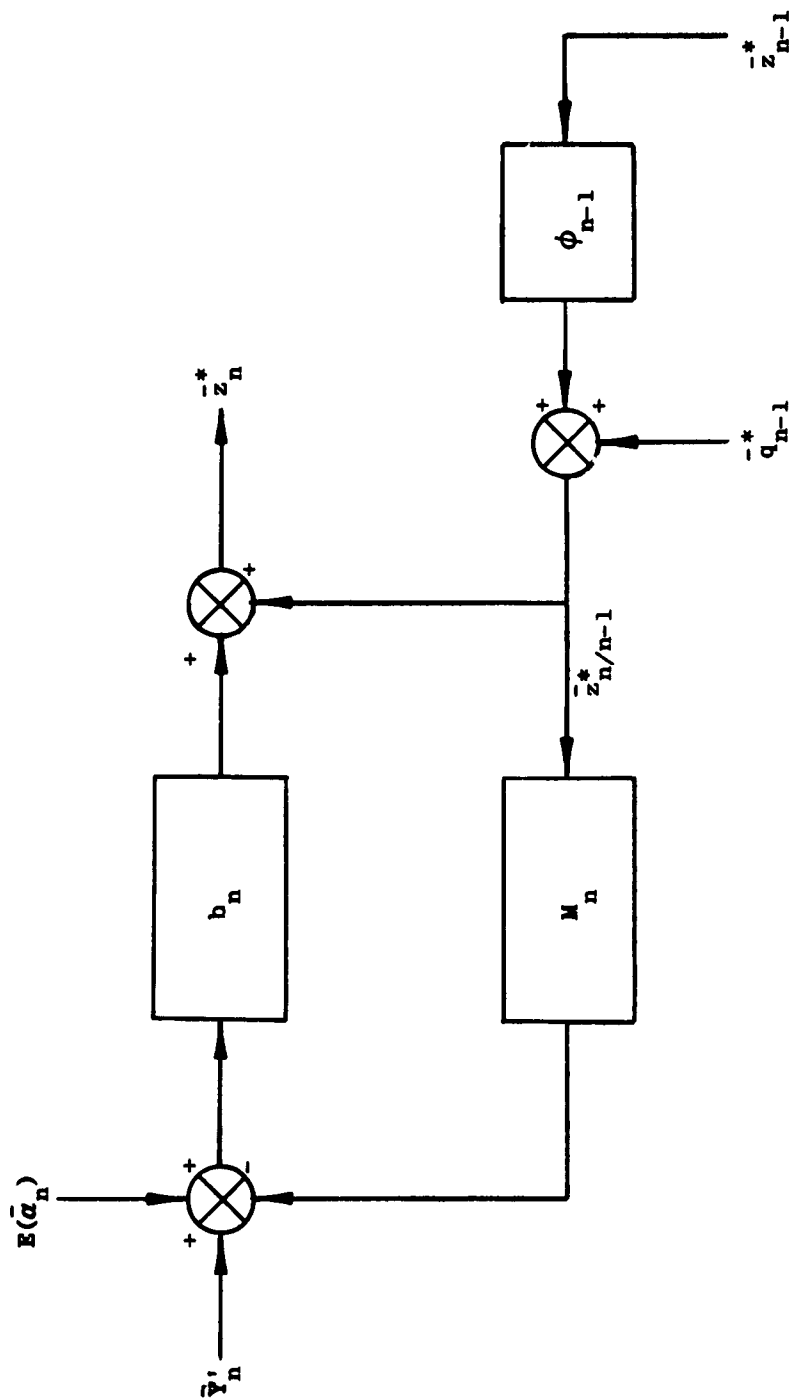
$$b_n = P_{n/n-1} M_n^T F_n^{-1}$$

The covariance for this estimate is

$$P_n^* = P_{n/n-1} - b_n F_n b_n^T$$

Thus, \bar{x}_n^* and P_n^* are known.

The block diagram of the optimal filter and estimator is shown in Figure 15.



$$P_{n/n-1} = \phi_{n-1}^* P_{n-1}^* \phi_{n-1}^T + E \left[(\bar{q}_{n-1}^* - \bar{q}_{n-1}) (\bar{q}_{n-1}^* - \bar{q}_{n-1})^T \right]$$

$$P_n^* = P_{n/n-1} - b_n^* F_n b_n^T$$

Figure 15. The Optimal Filter

5. THE OPTIMAL CONTROL LAW WITH IDENTIFICATION

5.1 General

The results of Chapters 3 and 4 are now combined in summary to achieve the complete optimization technique. The optimal estimates, assumed to exist in Chapter 3, are obtained by the Kalman filter in general form. That is, estimation for both the input and plant processes are combined into a single effort which reduces to appropriate conditions when specific conditions exist (e.g. deterministic inputs, etc.)

The condition for optimality utilizing these estimates is shown in the flow graphs for two alternative optimization procedures.

5.2 Summary of Equations and Optimization Scheme

The condition for optimal control is

$$\underset{(\bar{\rho}_{n-1}, \bar{\epsilon}_{n-1})}{\text{Minimize}} \left[F(\bar{\rho}_{n-1}, \sigma_{n-1}) \bar{\epsilon}_{n-1} - H_n^* \right] \quad (3.60)$$

where

$$F(\bar{\rho}, \sigma) = M^X \phi^X(\sigma) \int_0^{(\bar{\rho})_i} e^{-A\lambda} B \, d\lambda \quad (3.61)$$

$i=1, 2, \dots, q$

and

$$H_n^* = \bar{r}_{n/n-1}^* - M_n^X \phi_{n-1}^X \bar{x}_{n/n-1}^* \quad (3.53)$$

H_n^* represents the results of estimation and has been previously assumed to exist.

Investigating the significance of H_n^* , first consider the plant process defined in (2.10)

$$\bar{x}_n = \phi_{n-1}^X \bar{x}_{n-1} + A_{n-1} \bar{\epsilon}_{n-1} \quad (2.10)$$

$$\bar{c}_n = M_n^X \bar{x}_n$$

Assuming that no pulse is applied at $t = t_n$ ($A_{n-1} = 0$), and taking the conditional expectation of the first equation in (2.10), we get

$$\bar{x}_{n/n-1}^* = \phi_{n-1}^x \bar{x}_{n-1}^*$$

Multiplying by M_n^x ,

$$M_n^x \bar{x}_{n/n-1}^* = M_n^x \phi_{n-1}^x \bar{x}_{n-1}^* \quad (5.1)$$

This term then is the estimate of the output at $t = t_n$ when no pulse is applied at $t = t_{n-1}$. Since H_n^* is the difference between this term and the expected value of the reference input, it may be interpreted simply as the estimate of the future or extrapolated incremental error at $t = t_n$ when no pulse is applied at $t = t_{n-1}$. The pulse width and sign determined from the control law are chosen to minimize the incremental error that would occur at the next sampling instant if the pulse were not applied. A pulse exists then for the sole purpose of minimizing the effect of the nonexistence of that pulse. This is shown in Figure 16.

Since the estimation involves both the input and plant processes in the general case, a compact form using the stacked state vector \bar{z}_n will be used. Writing the function H_n^* in terms of \bar{z}_{n-1}^* , we have

$$H_n^* = -M_n^x \phi_{n-1}^x \bar{x}_{n-1}^* + M_n^w \phi_{n-1}^w \bar{w}_{n-1}^*$$

or

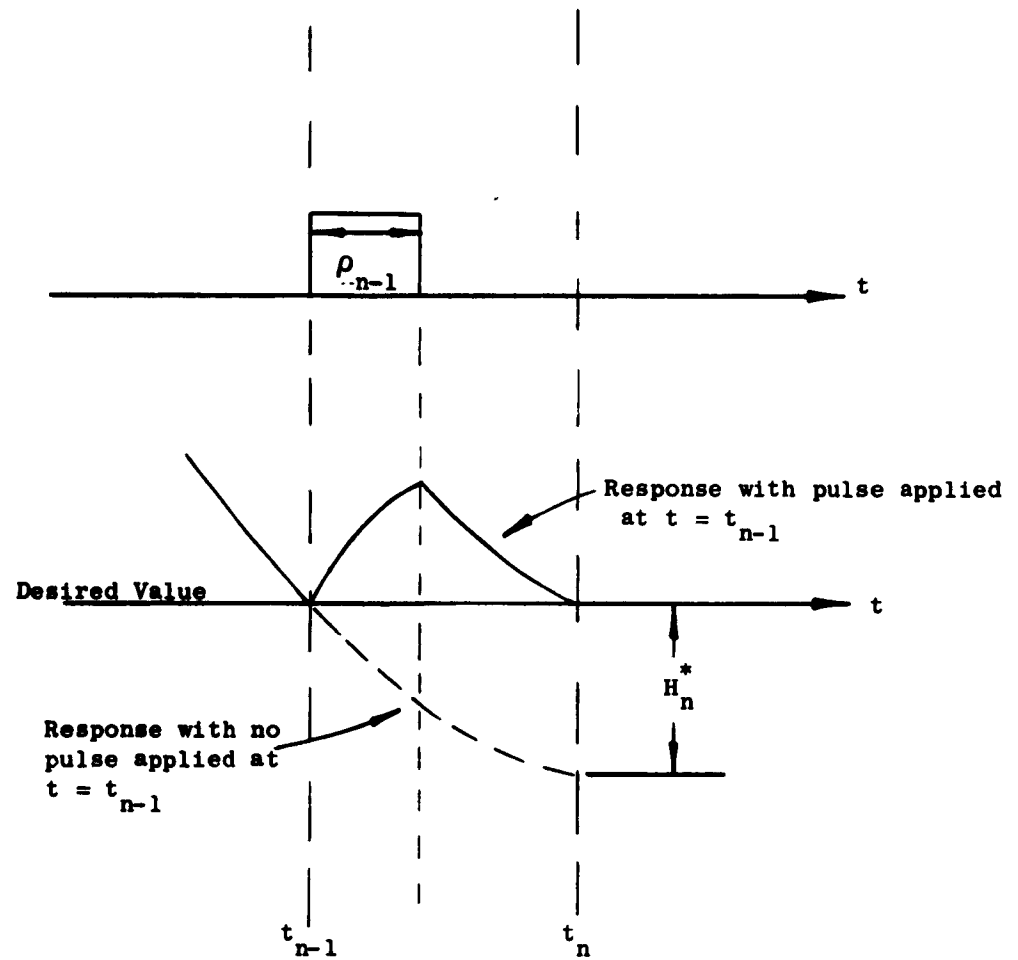
$$H_n^* = [-1 \quad 1] M_n \phi_{n-1} \bar{z}_{n-1}^*$$

The system constraint is

$$\begin{aligned} \bar{z}_n &= \phi_{n-1} \bar{z}_{n-1} + \Delta_{n-1} \bar{s}_{n-1} + \bar{A}_{n-1} \bar{e}_{n-1} \\ \bar{y}_n &= M_n \bar{z}_n \end{aligned} \quad (2.15)$$

The information constraint is included by considering measurements of the output corrupted by additive noise.

$$\bar{y}_n' = M_n \bar{z}_n + \bar{\alpha}_n \quad (5.2)$$



H_n^* = estimate of the incremental error at $t = t_n$ when no pulse is applied at $t = t_{n-1}$.

Figure 16. Use of Pulse to Cancel Future Error

The set of these measurements constitute the data to be used for estimation purposes at a particular sample time.

$$D_{n-1} = \{\bar{y}'_{n-1}, \bar{y}'_{n-2}, \dots, \bar{y}'_0\} \quad (5.3)$$

The filter equations are summarized as follows:

Begin at time $t = t_{n-1}$ and assume the knowledge of the optimal estimate of the state \bar{z}^*_{n-1} and the covariance of the estimation error P^*_{n-1} .

The predicted estimate of the state at $t = t_n$ is

$$\bar{z}^*_{n/n-1} = \phi_{n-1} \bar{z}^*_{n-1} + \bar{A}_{n-1} \bar{\epsilon}_{n-1} \quad (5.4)$$

The covariance in this estimate is

$$P_{n/n-1} = \phi_{n-1} P^*_{n-1} \phi_{n-1}^T + \Delta_{n-1} \Delta_{n-1}^T \quad (5.5)$$

Using the measurement \bar{y}'_n at $t = t_n$, the filtered estimate is

$$\bar{z}^*_n = (I - b_n M_n) \bar{z}^*_{n/n-1} - b_n \bar{y}'_n + b_n E(\bar{\alpha}_n) \quad (5.6)$$

where

$$F_n = M_n P_{n/n-1} M_n^T + \text{cov}(\bar{\alpha}_n, \bar{\alpha}_n) \quad (5.7)$$

and

$$b_n = P_{n/n-1} M_n^T F_n^{-1} \quad (5.8)$$

The covariance of the filtered estimate is

$$P^*_n = P_{n/n-1} - b_n F_n b_n^T \quad (5.9)$$

In writing equations (5.4) to (5.9), the following definitions have been made:

$$P_{n/n-1} = \begin{bmatrix} P^x_{n/n-1} & 0 \\ 0 & P^w_{n/n-1} \end{bmatrix}$$

$$b_n = \begin{bmatrix} b_n^x & 0 \\ 0 & b_n^w \end{bmatrix} \quad P_n^* = \begin{bmatrix} P_n^{*x} & 0 \\ 0 & P_n^{*w} \end{bmatrix}$$

$$F_n = \begin{bmatrix} F_n^x & 0 \\ 0 & F_n^w \end{bmatrix} \quad \bar{a}_n = \begin{bmatrix} \bar{a}_n^x \\ \bar{a}_n^w \end{bmatrix}$$

where

P is a $(m+v \times m+v)$ covariance matrix

P^x is a $(m \times m)$ covariance matrix for the plant state vector

P^w is a $(v \times v)$ covariance matrix for the input process state vector

F_n is a $(2p \times 2p)$ matrix

F_n^x is a $(p \times p)$ matrix for the plant process filter

F_n^w is a $(p \times p)$ matrix for the input process filter

b_n is a $(m+v \times 2p)$ matrix

b_n^x is a $(m \times p)$ matrix for the plant process filter

b_n^w is a $(v \times p)$ matrix for the input process filter

\bar{a}_n is a $(m+v \times 1)$ noise vector

\bar{a}_n^x is a $(m \times 1)$ noise vector for the plant process

\bar{a}_n^w is a $(v \times 1)$ noise vector for the input process

For the case of deterministic inputs or exact measurement of the state, the general filter equations reduce to the appropriate conditions. If, for example, the problem is completely deterministic, then $b_n = F_n = P_n = 0$ and the function H_n^* represents the exact incremental error involved.

The block diagram for the complete optimization procedure is shown in Figure 17.

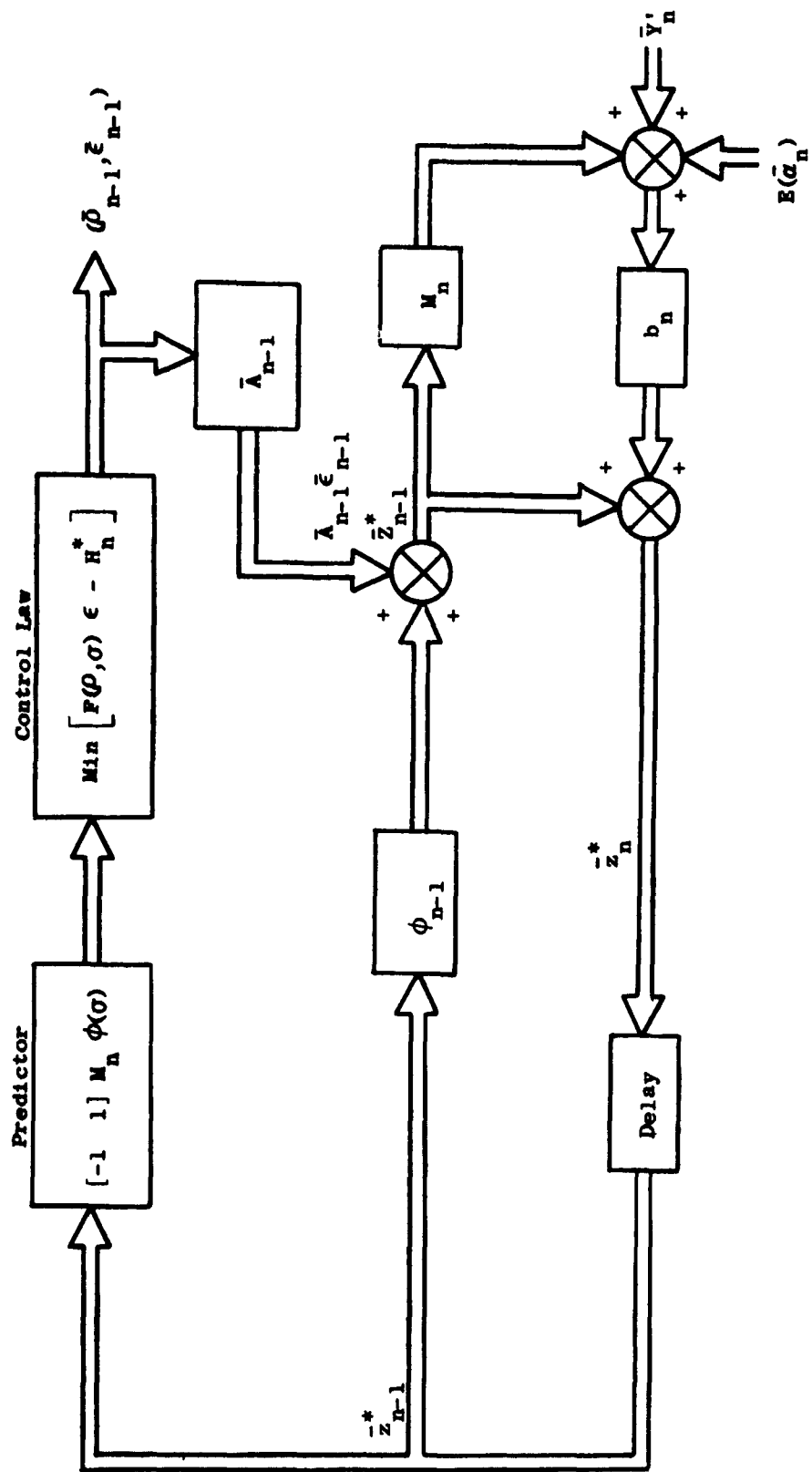


Figure 17. Block Diagram of Controller, Predictor and Filter

5.3 Alternate Scheme

For the program shown, there is a limitation due to the finite time required for the calculations involved. The pulse applied at time $t = t_n$ must be calculated using the data obtained at that same time. The optimal estimate must be found and the procedure for determining the optimum pulse width must be executed before the pulse may actually be applied. The time required for these calculations delays the application of the pulse sufficiently in some cases. If this limitation is too severe due to high sensitivity of the output with respect to pulse width, or the system is operated with a fast sampling rate compared to the time of calculation, then an alternate procedure must be developed.

The alternate procedure is quite simple in that predicted estimates rather than filtered estimates are used where the prediction and other calculations are executed throughout the sampling interval. That is, at time $t = t_n$, a measurement of the output is available, but will be used for predicting the state at $t = t_{n+1}$. The function H_{n+1}^* is determined from this predicted value and the optimal pulse width and sign are then calculated. These operations are performed during the sampling interval and the pulse to be applied at the next sampling instant is ready. This alternate procedure limits the calculation time to the length of the sampling interval, but the obvious extension of predicting several samples ahead may be used in cases of this nature.

The flow graph for the alternate scheme is shown in Figure 18.

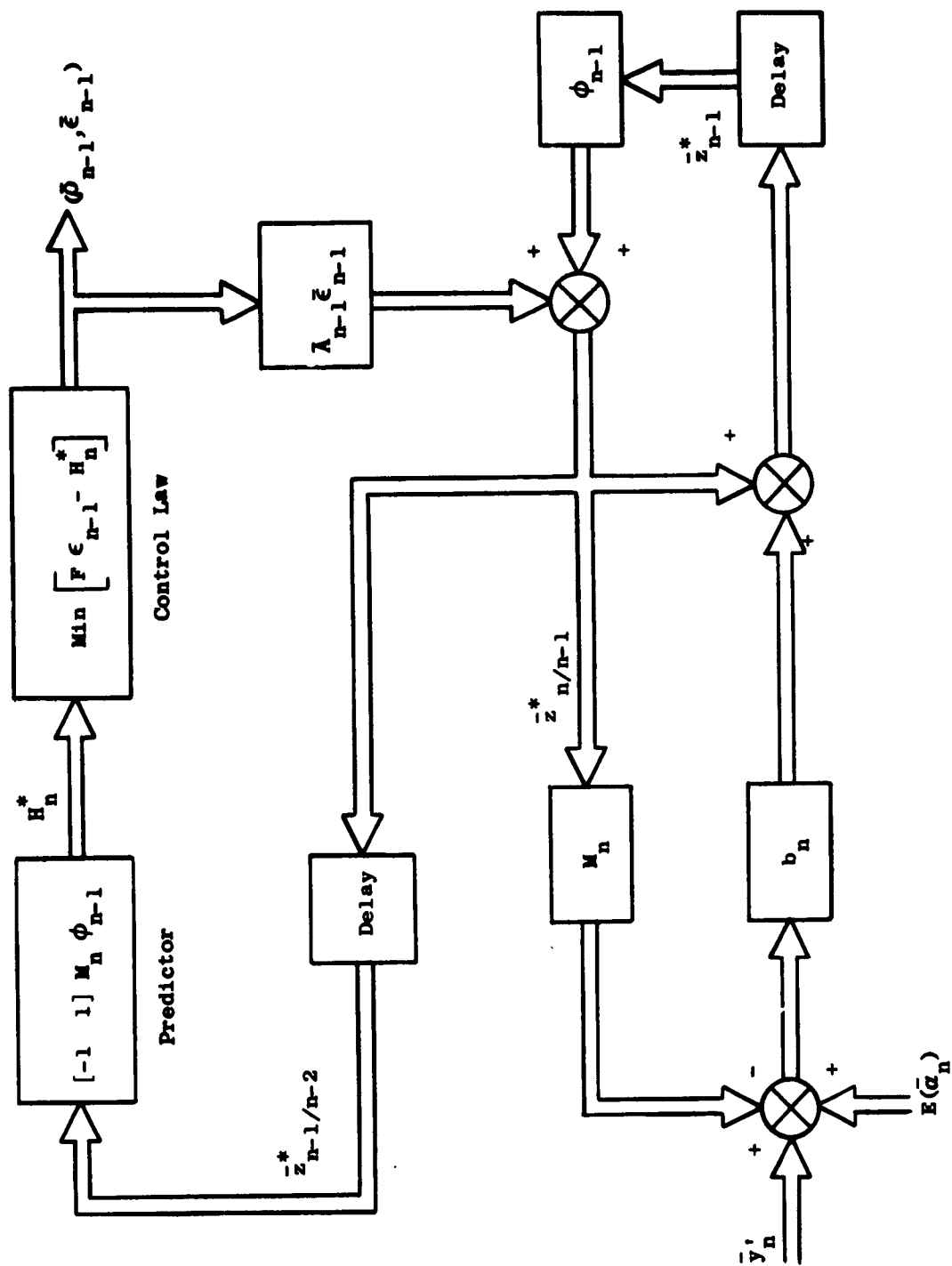


Figure 18. Block Diagram of Alternate Scheme

6. EXAMPLES

6.1 General

This chapter is intended to illustrate the optimization technique described in the previous chapters. It is hoped that the validity and usefulness of the theory will be demonstrated at the same time. Numerical results have been obtained through simulation of the entire system on a high-speed digital computer.

The computer involved in this simulation was the Control Data Corporation 1604. The entire program was written using the FORTRAN compiler language. The extensively tested³⁵ normal deviates from the table "A MILLION RANDOM DIGITS WITH 100,000 NORMAL DEVIATES"³⁶ were transcribed in part on magnetic tape and used as the source of additive noise in all cases. The program included the choice of mean and covariance for these normal deviates. A random number generator, which was programmed in machine language, together with von Neumann's rejection technique³⁷ for sampling from various other distributions was also available.

The optimal response is compared to a normal response in each case where normal pulse width control refers to a standard procedure used to determine the pulse width and sign at each sampling instant. This standard procedure is sometimes called "lead" pulse width modulation where the width and sign of each pulse are directly related to the magnitude and sign of the actuating signal at the sampling instant.

$$\rho_n = a |e_n|$$

$$\epsilon_n = \text{sign of } (e_n)$$

The parameter "a" was taken as unity for convenience.

The block diagram of the system under consideration for each example is shown in Figure 19.

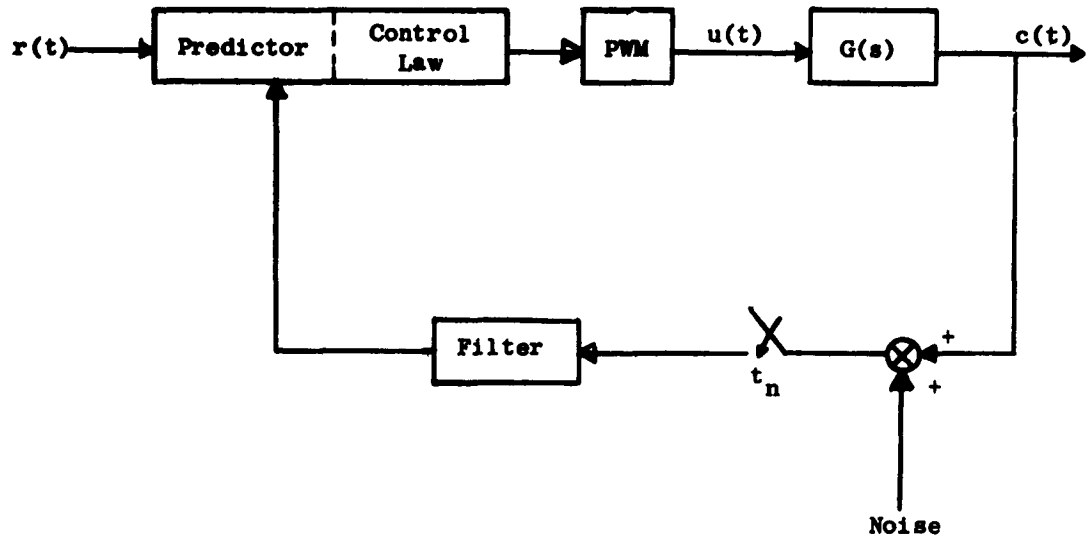


Figure 19. Block Diagram of Simulated Systems

6.2 A Second Order System

The linear plant for this example is described by the transfer function

$$G(s) = \frac{K}{s(s+1)} = \frac{C(s)}{U(s)}$$

Using state vector notation, the plant process may be written

$$\dot{\bar{x}} = A \bar{x} + B \bar{u}$$

$$\bar{c} = M \bar{x}$$

$$A = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ K \end{bmatrix} \quad M = [1 \quad 0]$$

The discrete form is

$$\bar{x}_{n+1} = \phi_n^x \bar{x}_n + A_n \bar{e}_n$$

$$\bar{c}_n = M_n \bar{x}_n$$

where

$$\phi^x(t) = e^{At} = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$Y = \int_0^\rho e^{-At} B dt = \begin{bmatrix} \rho - (e^\rho - 1) \\ (e^\rho - 1) \end{bmatrix}$$

$$A(\rho, \sigma) = \phi^x(\sigma) Y = \begin{bmatrix} \rho - e^{-\sigma}(e^\rho - 1) \\ e^{-\sigma}(e^\rho - 1) \end{bmatrix}$$

The step responses for plant gains of 4.0 and 5.0 are shown in Figures 20 and 21 for both exact and noisy measurements of the state. The additive noise was assumed normally distributed with zero mean and unity covariance. The sampling rate was assumed constant at 0.1 seconds.

The normal pulse width response was found to be unstable for any gain greater than 5.0. The optimum system was found to be stable for gains much higher than 5.0 for both step and ramp inputs. The response for a gain of 10.0 is shown in Figure 22.

The ramp responses for these gains are shown in Figure 23.

The results show a marked improvement in performance for these inputs. For the step response, this is attributed to the small pulse applied before the desired value is reached allowing the system to coast for the remainder of the sampling interval and then settle rapidly thereafter. This of course is due entirely to the minimization of the future errors.

The flow graph for a digital controller is shown in Figure 24. The analog realization is shown in Figure 25.

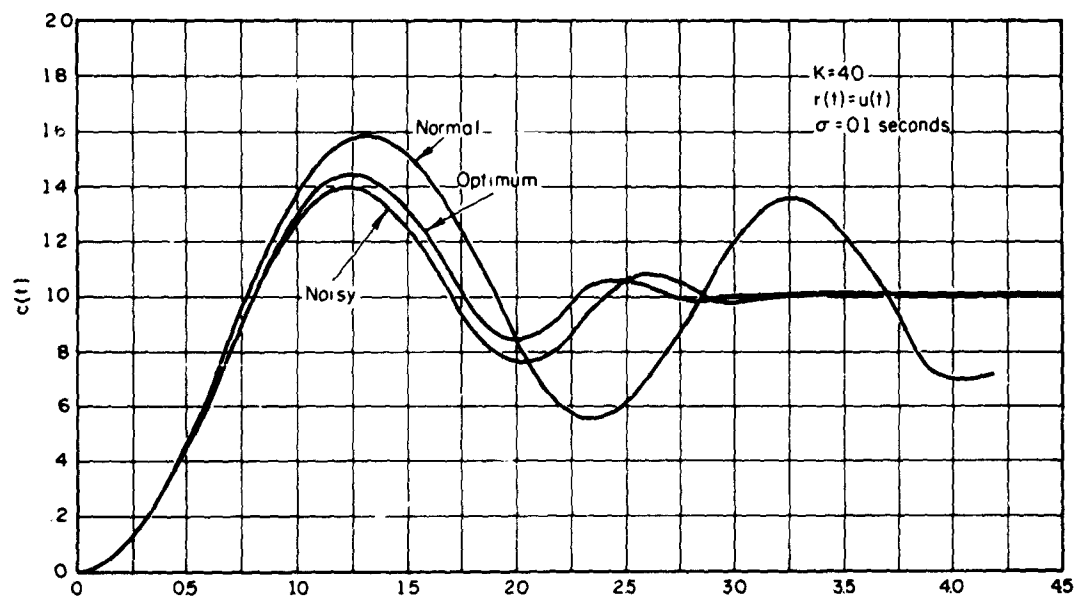


Figure 20. Step Response of Second Order System $G(s) = \frac{K}{s(s+1)}$

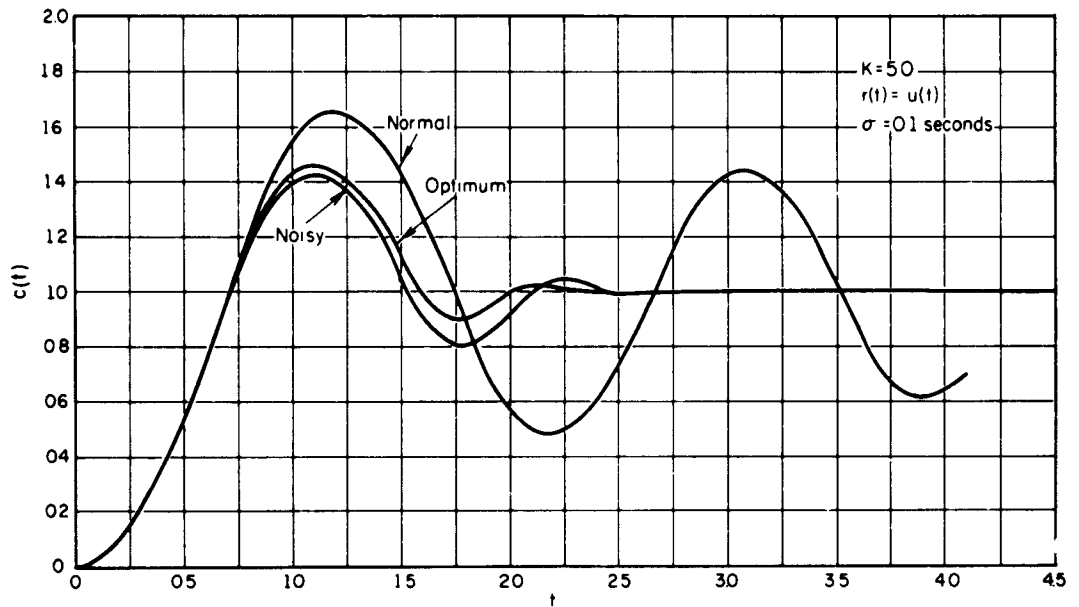


Figure 21. Step Response of Second Order System $G(s) = \frac{K}{s(s+1)}$

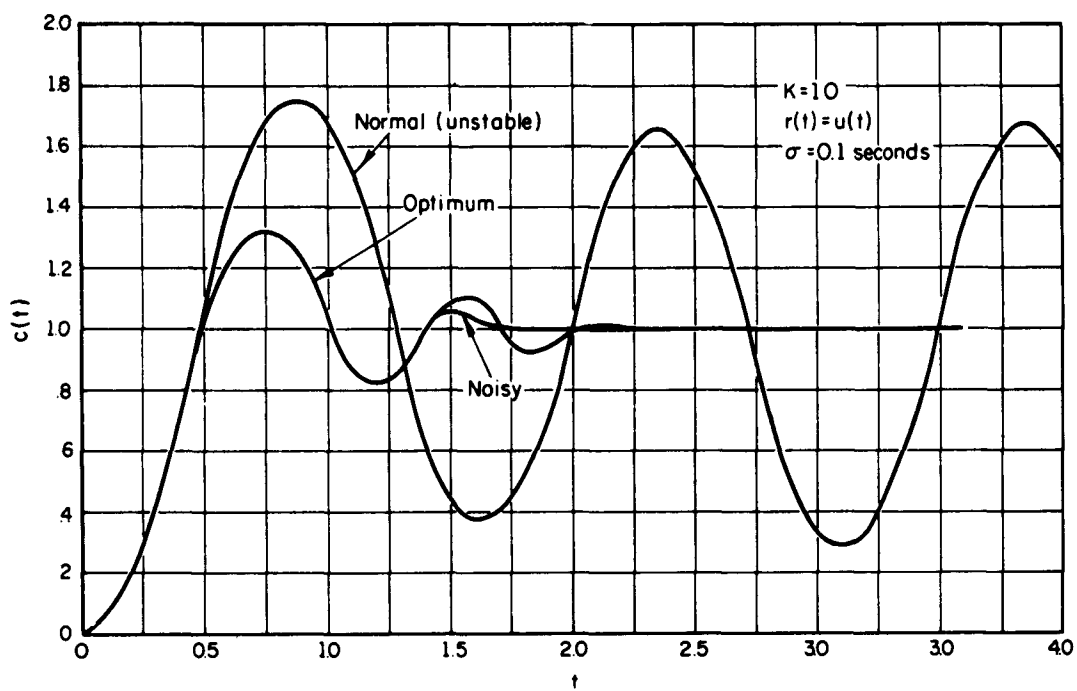


Figure 22. Step Response of Second Order System $G(s) = \frac{K}{s(s+1)}$

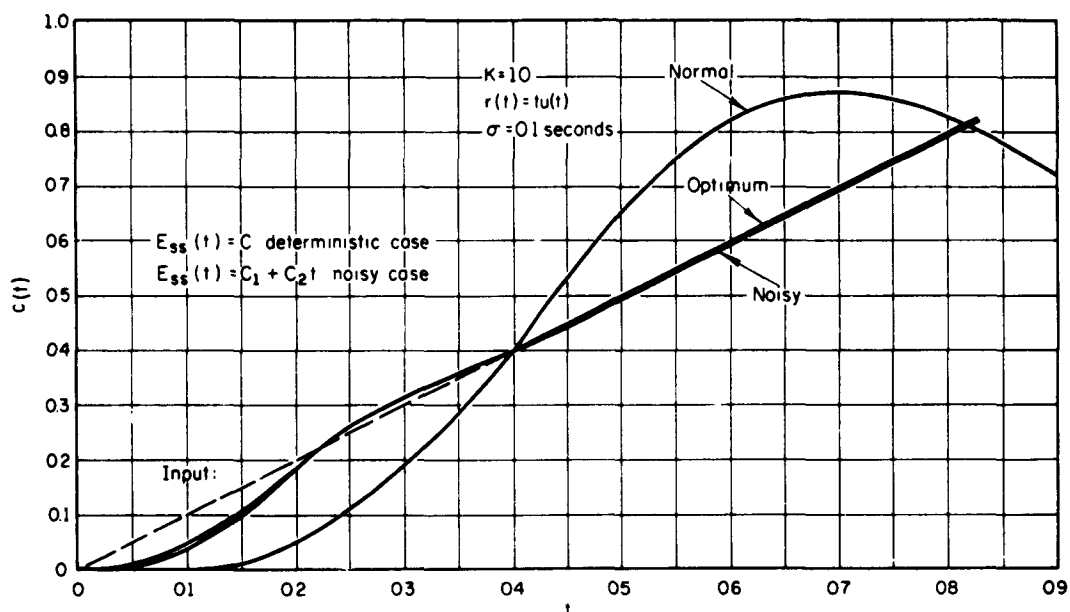


Figure 23. Ramp Response of Second Order System $G(s) = \frac{K}{s(s+1)}$

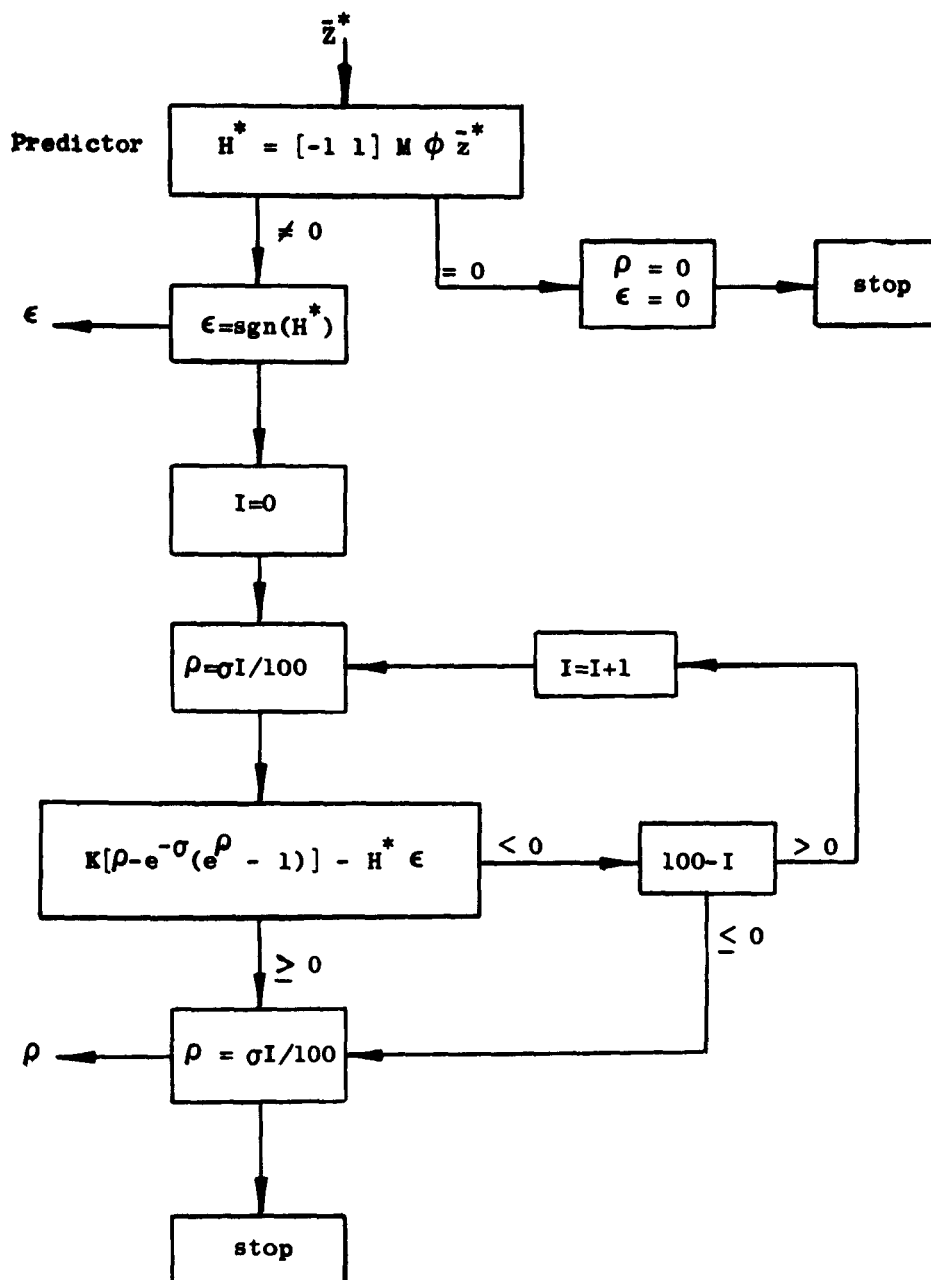


Figure 24. Flow Graph of Controller and Predictor Program for Second Order System

$$G(s) = \frac{K}{s(s+1)}$$

6.3 A Third Order System

A third order system with complex roots is investigated in this section. The transfer function of the plant is

$$G(s) = \frac{K}{s(s^2 + 2s + 2)} \quad (6.3)$$

The block diagram of the system is shown in Figure 19. Preliminary calculations lead to the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} \quad M = [1 \quad 0 \quad 0]$$

$$\phi^x(t) = e^{At} = \begin{bmatrix} 1 & 1 - e^{-t} \cos(t) & \frac{1}{2} - \frac{\sqrt{2}}{2} e^{-t} \sin(t+45^\circ) \\ 0 & \sqrt{2} e^{-t} \sin(t+45^\circ) & e^{-t} \sin(t) \\ 0 & -2 e^{-t} \sin(t) & -\sqrt{2} e^{-t} \sin(t-45^\circ) \end{bmatrix}$$

$$Y = \int_0^\rho e^{-At} B dt = \begin{bmatrix} \frac{\rho}{2} - \frac{1}{2} e^\rho \cos(\rho) + \frac{1}{2} \\ -\frac{1}{2} + \frac{\sqrt{2}}{2} e^\rho \cos(\rho+45^\circ) \\ e^\rho \sin(\rho) \end{bmatrix}$$

$$A(\rho, \sigma) = \phi^x(\sigma) Y$$

A plant gain of 2.0 and a constant sampling rate of 1.20 seconds was used for this example. The unit step response is shown for exact and noisy measurements of the state in Figure 26.

Since a nonlinear system is under investigation, the unit step response has very little significance in terms of describing the system performance to

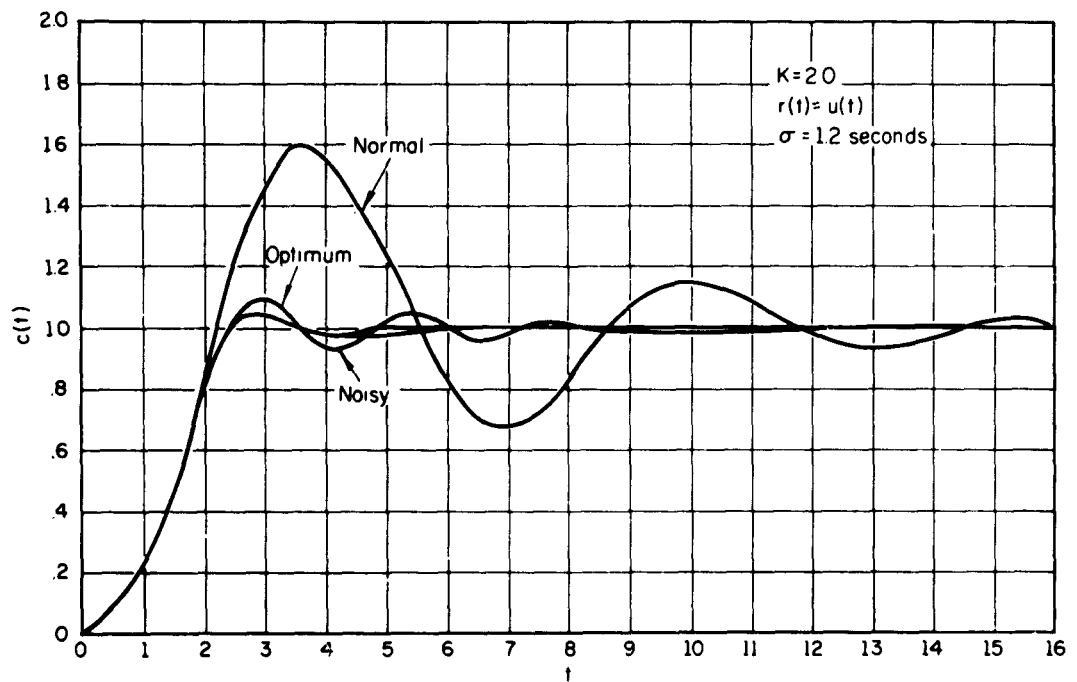


Figure 26. Step Response of Third Order System $G(s) = \frac{K}{s(s^2 + 2s + 2)}$

displacement inputs. For this reason, a family of step responses is shown in Figure 27 for the optimal mode. The corresponding normal responses are shown in Figure 28. A direct comparison is shown in Figure 29.

For large step inputs the system approaches the behavior of the normal system since the on-off mode of operation is in effect in both cases until just before settling time. Also, since the system response time is very slow for large inputs, little information will be gained by considering these large displacement inputs.

For small step inputs, the behavior depends a great deal on the sampling rate and begins to exhibit rather sporadic results for large sampling rates compared to the system response time. The criterion for optimality is satisfied in all cases, but it becomes doubtful whether or not the criterion has any meaning in relation to what we might consider good response with respect to our aesthetic criterion for optimal behavior. A good example of this is seen for the 0.70 step response in the optimal mode. The system eventually settles and satisfies the criterion established, but our judgment dictates that the normal response in this case is better with respect to our aesthetic optimality criterion. Thus, the optimal mode leads to undesirable results for some ranges of operation. The pole-zero location in relation to the magnitude of input information for various sampling rates is seen to be an important consideration when using this optimization technique since there are desirable and undesirable ranges of operation.

No attempt was made to investigate thoroughly the change of stability boundaries with respect to gain and sampling rate but there seems to be an extremely interesting effect involved. For a gain of 3.0 and a unit sampling rate, the system is stable in the optimal mode but unstable in the normal mode. On the other hand, for a gain of 2.0 and a unit sampling rate, the reverse is true. The explanation for this may be tied in with the fact that these points lie very near the stability boundary.

For a ramp input, the system exhibited a constant steady state positional error of 2.20 units in the normal mode when the state was known exactly. In the optimal mode for exact measurement of the state, the steady state positional error was improved to 1.0 units. In the optimal mode when the information about

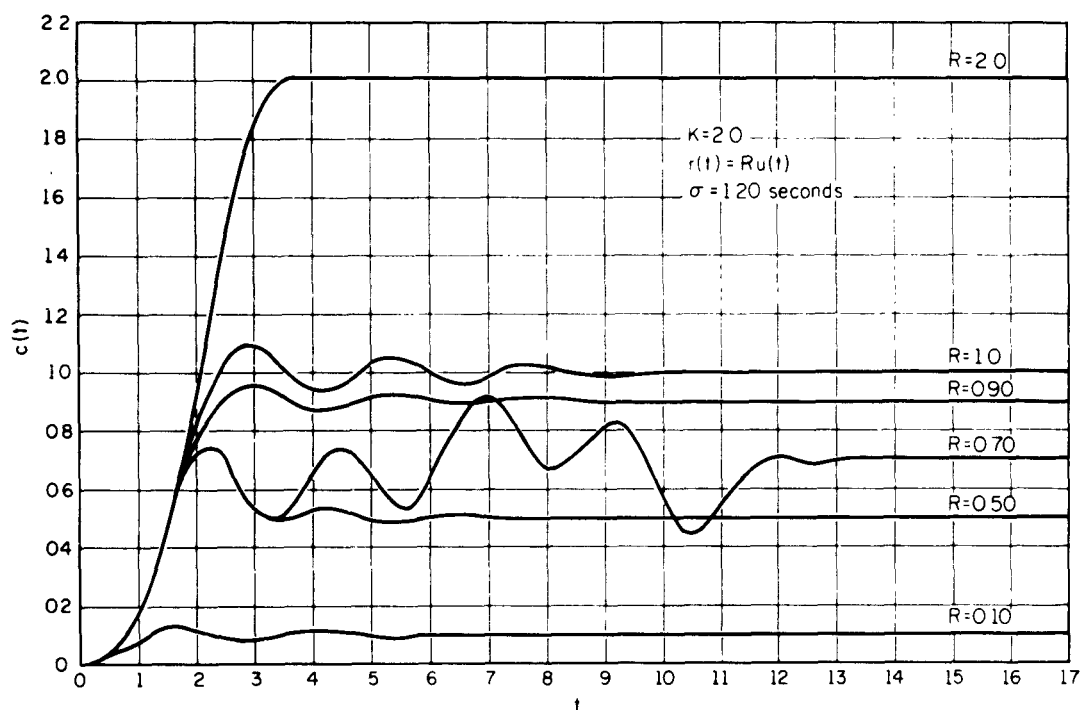


Figure 27. Family of Optimal Step Responses for Third Order System

$$G(s) = \frac{K}{s(s^2 + 2s + 2)}$$

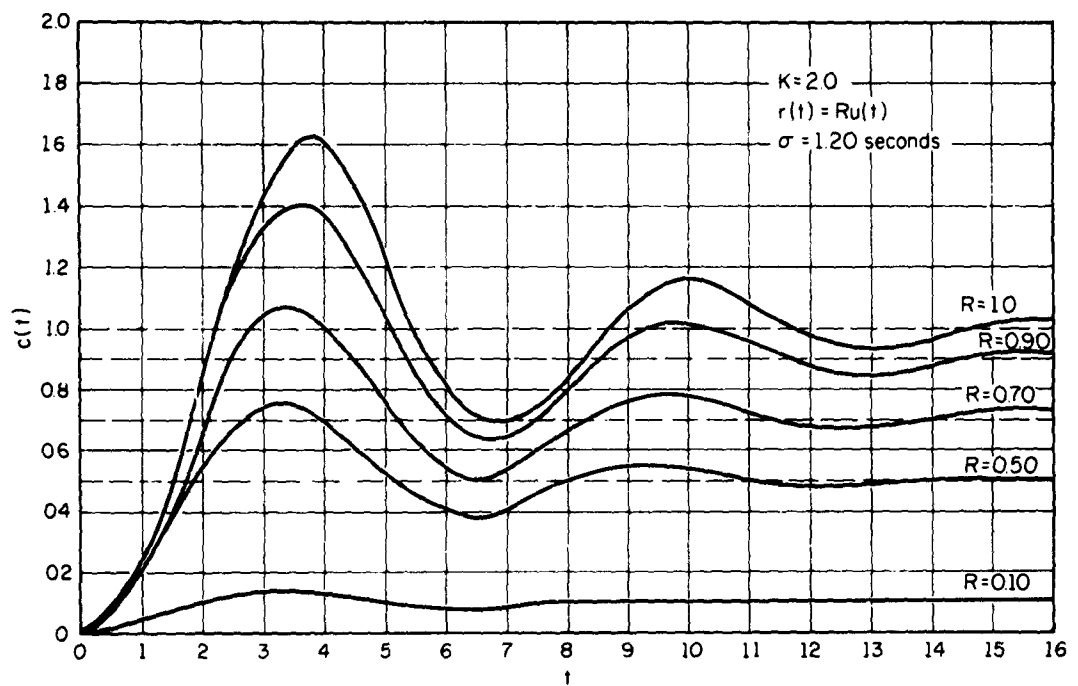


Figure 28. Family of Step Responses for Third Order System

$$G(s) = \frac{K}{s(s^2 + 2s + 2)}$$

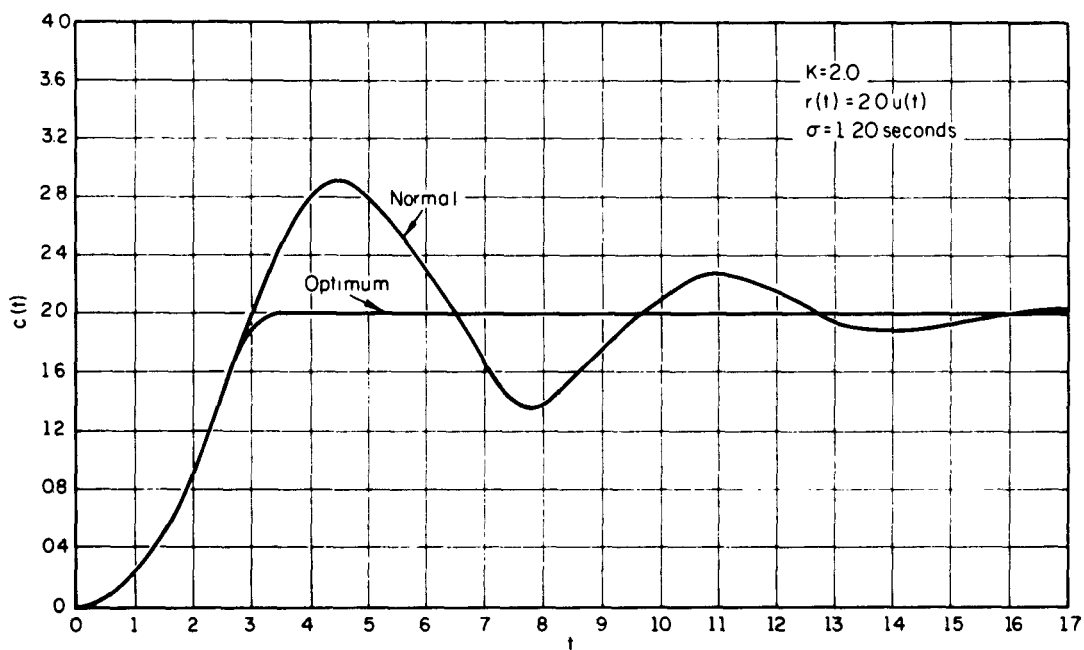


Figure 29. Step Response of Third Order System $G(s) = \frac{K}{s(s^2 + 2s + 2)}$

the state was constrained, the steady state error was found to be a linear function of time. These results are shown in Figure 30. The steady state error series for each case is listed as follows:

Normal mode; exact measurement of the state

$$E_{ss}(t) = 2.20$$

Optimal mode; exact measurement of the state

$$E_{ss}(t) = 1.00$$

Optimal mode; noisy measurement of the state

$$E_{ss}(t) = 0.346 t - 6.5$$

6.4 A Fourth Order System with Distinct Roots

A fourth order system with a zero is investigated in this section. The transfer function of the plant is

$$G(s) = \frac{K(s+2)}{s(s+1)(s+3)(s+4)} \quad (6.4)$$

The block diagram of the system is again in Figure 19. Preliminary calculations of the necessary matrices lead to the following results:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -12 & -19 & -8 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ K \\ -6K \end{bmatrix}$$

$$M = [1 \quad 0 \quad 0 \quad 0]$$

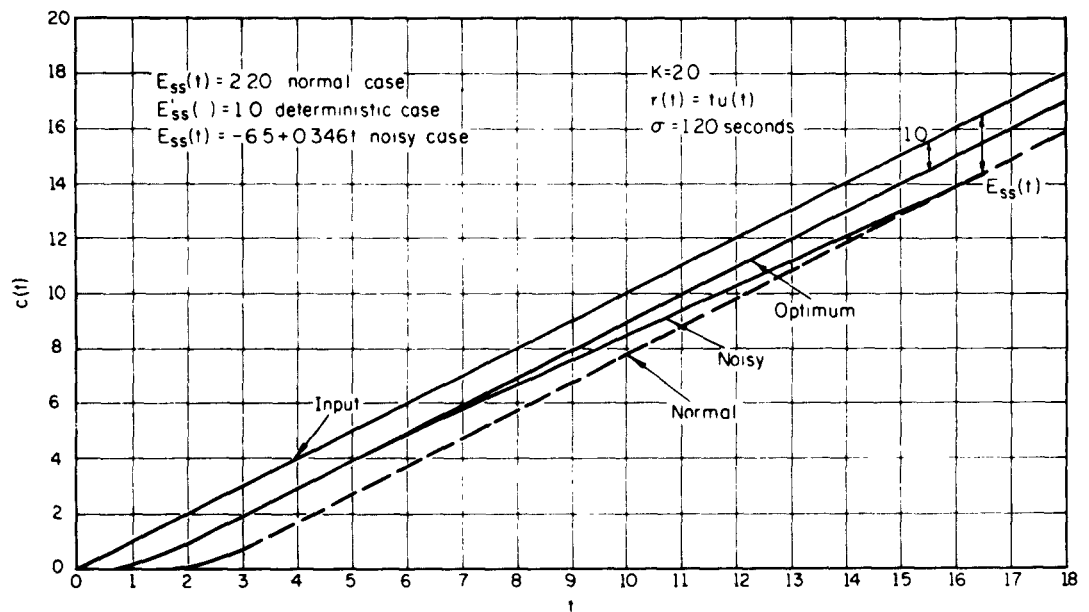


Figure 30. Ramp Response of Third Order System $G(s) = \frac{K}{s(s^2 + 2s + 2)}$

$$\phi(t) = e^{At} = \begin{bmatrix} 1 & \frac{19}{12} - 2e^{-t} + \frac{2}{3}e^{-3t} - \frac{1}{4}e^{-4t} & \frac{2}{3} - \frac{7}{6}e^{-t} + \frac{5}{6}e^{-3t} - \frac{1}{3}e^{-4t} & \frac{1}{12} - \frac{1}{16}e^{-t} + \frac{1}{6}e^{-3t} - \frac{e^{-4t}}{12} \\ 0 & 2e^{-t} - 2e^{-3t} + e^{-4t} & \frac{7}{6}e^{-t} - 2.5e^{-3t} + \frac{4}{3}e^{-4t} & \frac{1}{6}e^{-t} - 0.5e^{-3t} + \frac{1}{3}e^{-4t} \\ 0 & -2e^{-t} + 6e^{-3t} - 4e^{-4t} & -\frac{7}{6}e^{-t} + \frac{15}{2}e^{-3t} + \frac{16}{3}e^{-4t} & -\frac{1}{6}e^{-t} + 1.5e^{-3t} - \frac{4}{3}e^{-4t} \\ 0 & 2e^{-t} - 18e^{-3t} + 16e^{-4t} & \frac{7}{6}e^{-t} - 22.5e^{-3t} + \frac{64}{3}e^{-4t} & \frac{1}{6}e^{-t} - \frac{9}{2}e^{-3t} + \frac{16}{3}e^{-4t} \end{bmatrix}$$

$$Y = \begin{bmatrix} \frac{\rho}{6} - \frac{1}{6}(e^{\rho} - 1) - \frac{1}{18}(e^{3\rho} - 1) + \frac{1}{24}(e^{4\rho} - 1) \\ \frac{1}{6}(e^{\rho} - 1) + \frac{1}{6}(e^{3\rho} - 1) - \frac{1}{6}(e^{4\rho} - 1) \\ -\frac{1}{6}(e^{\rho} - 1) - 0.5(e^{3\rho} - 1) + \frac{2}{3}(e^{4\rho} - 1) \\ \frac{1}{6}(e^{\rho} - 1) + 1.50(e^{3\rho} - 1) - \frac{8}{3}(e^{4\rho} - 1) \end{bmatrix}$$

The characteristic matrix for higher order systems is somewhat involved but may be calculated easily using Mason's gain formula as presented by B.C. Kuo³⁴.

A family of step responses is shown in Figure 31 for a gain of 6.0 and a unit sampling rate. Again there is a range of inputs which lead to results which may be undesirable. In particular, the input step of 0.3 units begins following the 0.4 unit input response until about 2.5 seconds resulting in a large initial overshoot. This result of course depends on the sampling rate chosen and is expected in this case since the rate is quite low compared to the settling time for this lower input level. In general, we would expect unusual results when the sampling rate is larger than the system response or settling time.

In order to compare the normal and optimal responses, a quality factor which seems to be reasonable is the 5% settling time for the two systems. Figure 32 shows the settling time versus the magnitude of displacement input. For large inputs the on-off mode is predominant and the two curves approach the same asymptote as shown in the figure. For input magnitudes greater than 40.0, the settling time curves for both systems are within 5% of the asymptote.

Investigating the behavior of the two systems further, another quality factor examined was the first overshoot versus input magnitude. This curve is shown in Figure 33 and we see immediately that lower level inputs exhibit rather sporadic first overshoots due primarily to the large sampling rate. Note, however, that in this case even though the first overshoot is larger than that for the normal system for an input of 0.3 units, the settling time is still about 60% lower than the normal response.

Since the sampling rate is fixed at unity, it is not to our advantage here to consider step responses for input signal levels less than 0.1 units. A family of responses for various other sampling rates and input levels would be in order however, if the technique were to be applied to existing systems.

6.5 A Fourth Order System with Complex Roots

A fourth order system with complex roots is investigated in this section. The transfer function of the plant is

$$G(s) = \frac{K}{s(s+3)(s^2 + 2s + 2)} = \frac{C(s)}{U(s)} \quad (6.5)$$

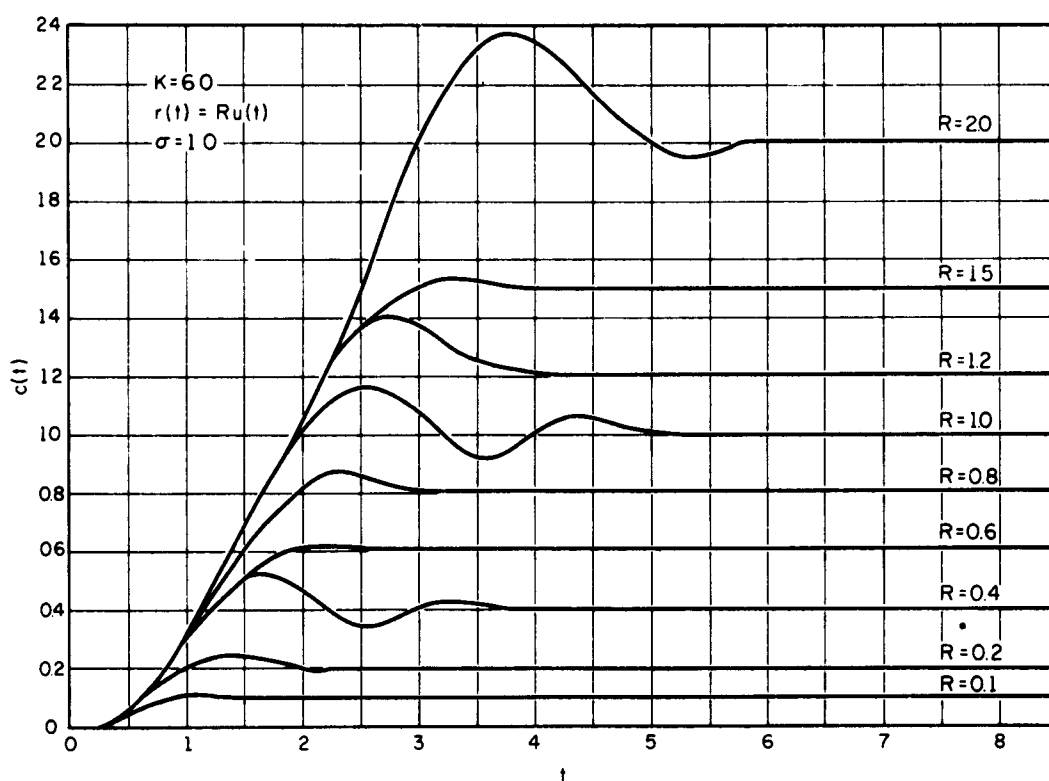


Figure 31. Family of Optimal Step Responses for Fourth Order System with Distinct Roots

$$G(s) = \frac{K(s+2)}{s(s+1)(s+3)(s+4)}$$

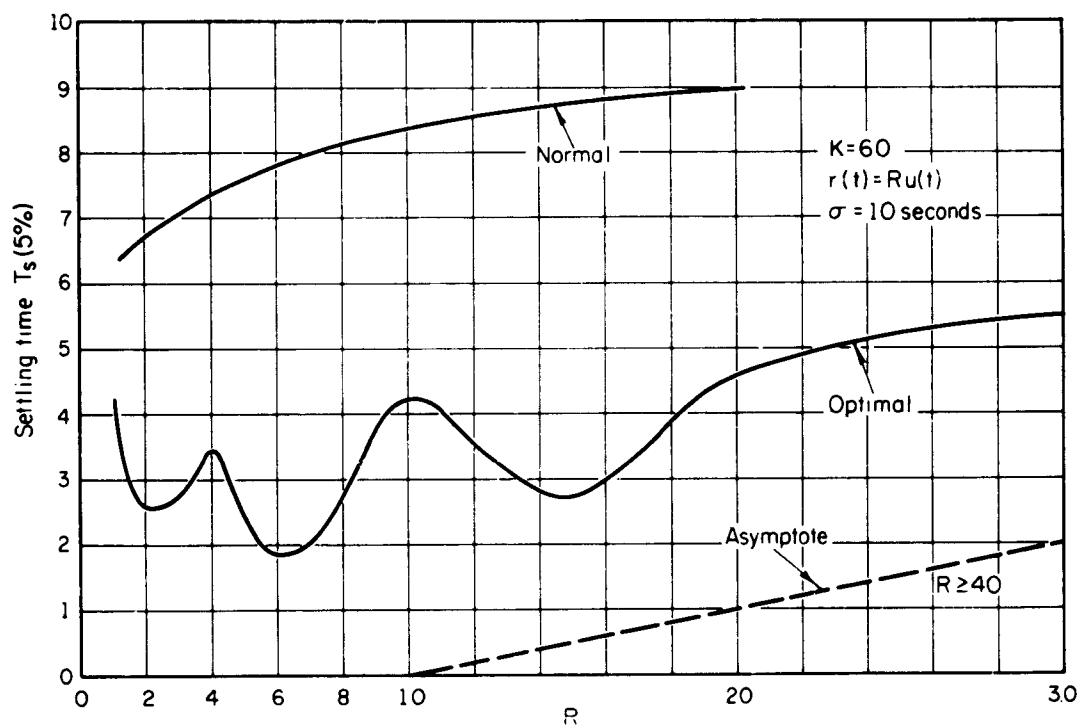


Figure 32. Settling Time Versus Magnitude of Displacement Input for Fourth Order System with Distinct Roots

$$G(s) = \frac{K(s+2)}{s(s+1)(s+3)(s+4)}$$

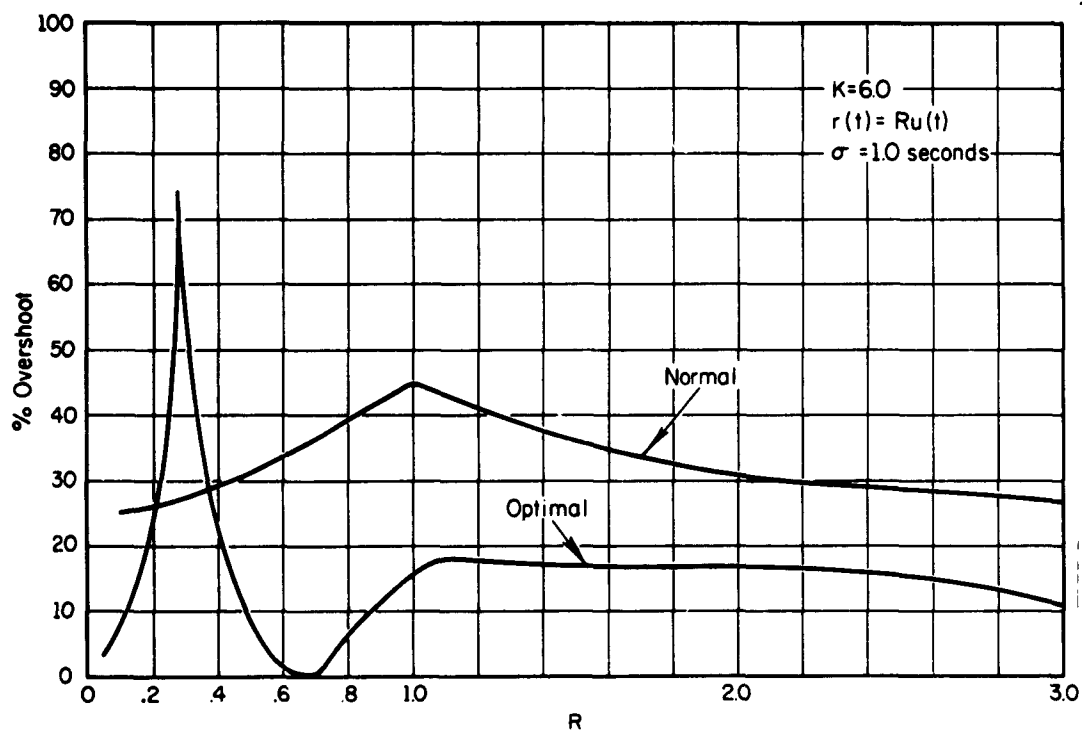


Figure 33. Per Cent Overshoot Versus Magnitude of Displacement Input for Fourth Order System with Distinct Roots

$$G(s) = \frac{K(s+2)}{s(s+1)(s+3)(s+4)}$$

The initial calculations lead to the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -6 & -8 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix}$$

$$M = [1 \quad 0 \quad 0 \quad 0]$$

The elements of the characteristic matrix $\phi(t) = e^{At}$ are:

$$\phi_{11}(t) = 1$$

$$\phi_{12}(t) = \frac{4}{3} - \frac{2}{15} e^{-3t} + \frac{6\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) - \frac{9}{5} e^{-t} \sin(t)$$

$$\phi_{13}(t) = \frac{5}{6} - \frac{2}{15} e^{-3t} + \frac{7\sqrt{2}}{10} e^{-t} \sin(t-45^\circ) - \frac{9}{5} e^{-t} \sin(t)$$

$$\phi_{14}(t) = \frac{1}{6} - \frac{1}{15} e^{-3t} + \frac{\sqrt{2}}{10} e^{-t} \sin(t-45^\circ) - \frac{2}{5} e^{-t} \sin(t)$$

$$\phi_{21}(t) = 0$$

$$\phi_{22}(t) = \frac{2}{5} e^{-3t} - \frac{3\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) + \frac{12}{5} e^{-t} \sin(t)$$

$$\phi_{23}(t) = \frac{2}{5} e^{-3t} + \frac{2\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) + \frac{7}{5} e^{-t} \sin(t)$$

$$\phi_{24}(t) = \frac{1}{5} e^{-3t} + \frac{\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) + \frac{1}{5} e^{-t} \sin(t)$$

$$\phi_{31}(t) = 0$$

$$\phi_{32}(t) = -\frac{6}{5} e^{-3t} - \frac{6\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) - \frac{6}{5} e^{-t} \sin(t)$$

$$\phi_{33}(t) = -\frac{6}{5} e^{-3t} - \frac{11\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) + \frac{4}{5} e^{-t} \sin(t)$$

$$\phi_{34}(t) = -\frac{3}{5} e^{-3t} - \frac{3\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) + \frac{2}{5} e^{-t} \sin(t)$$

$$\phi_{41}(t) = 0$$

$$\phi_{42}(t) = \frac{18}{5} e^{-3t} + \frac{18\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) - \frac{12}{5} e^{-t} \sin(t)$$

$$\phi_{43}(t) = \frac{18}{5} e^{-3t} + \frac{18\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) - \frac{22}{5} e^{-t} \sin(t)$$

$$\phi_{44}(t) = \frac{9}{5} e^{-3t} + \frac{4\sqrt{2}}{5} e^{-t} \sin(t-45^\circ) - \frac{6}{5} e^{-t} \sin(t)$$

$$Y = \begin{bmatrix} \frac{1}{5} + \frac{\rho}{6} - \frac{1}{45}(e^{3\rho} - 1) - \frac{1}{10} e^{\rho} \sin(\rho) - \frac{\sqrt{2}}{5} e^{\rho} \cos(\rho+45^\circ) \\ -\frac{1}{10} + \frac{1}{15}(e^{3\rho} - 1) - \frac{1}{5} e^{\rho} \sin(\rho) + \frac{\sqrt{2}}{10} e^{\rho} \cos(\rho+45^\circ) \\ -\frac{1}{5} - \frac{1}{5}(e^{3\rho} - 1) + \frac{3}{5} e^{\rho} \sin(\rho) + \frac{\sqrt{2}}{5} e^{\rho} \cos(\rho+45^\circ) \\ \frac{3}{5} + \frac{3}{5}(e^{3\rho} - 1) - \frac{4}{5} e^{\rho} \sin(\rho) - \frac{3\sqrt{2}}{5} e^{\rho} \cos(\rho+45^\circ) \end{bmatrix}$$

The step response curves for a gain of 5.0 and a sampling rate of 1.40 seconds are shown in Figure 34. The input level in this case is 2.0 units. In Figure 35 the response is shown for an input signal level of 0.8 units.

6.6 Comments

From the several examples presented here, we see that the system performance is greatly improved for certain ranges of gains, sampling rates and input signal levels when optimized using the technique presented in the previous chapters. There are, of course, ranges where the response satisfies the criterion established but where it does not satisfy some aesthetic criterion for optimal behavior. In particular, for the families of curves shown, the sampling rate was assumed constant and for low level signal inputs, the response time of the system is smaller than the sampling rate leading to results that may be classified as undesirable. The results demonstrate, however, the ability of the system to achieve optimal performance with respect to the established criterion in all cases and excellent performance not otherwise possible in many cases.

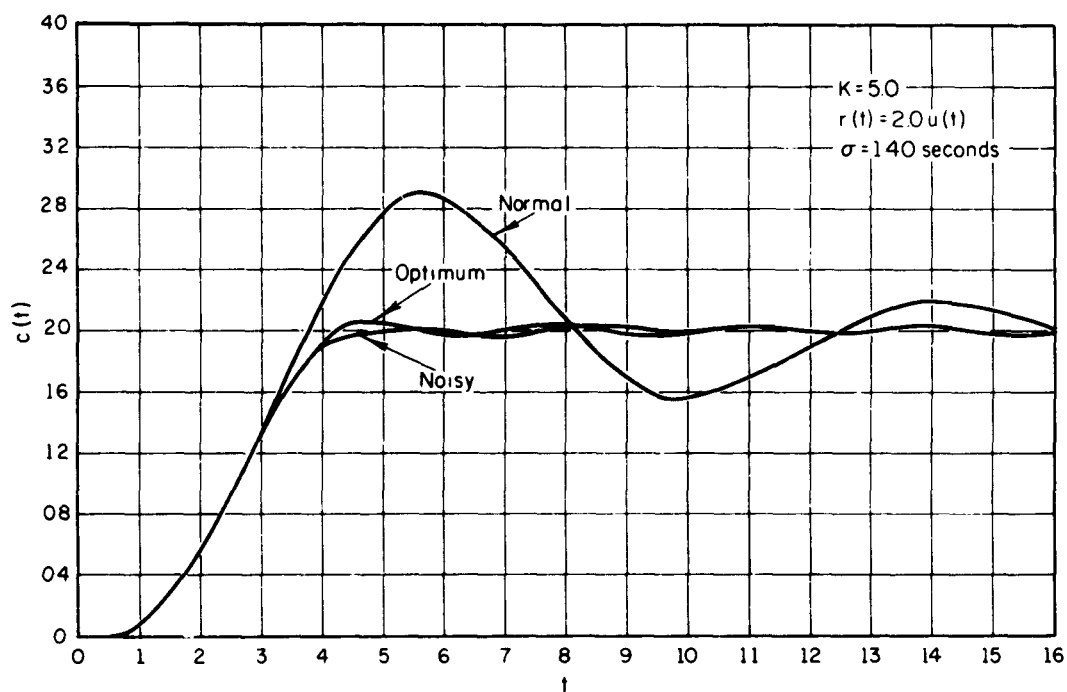


Figure 34. Step Response of Fourth Order System with Complex Roots

$$G(s) = \frac{K}{s(s+3)(s^2 + 2s + 2)}$$

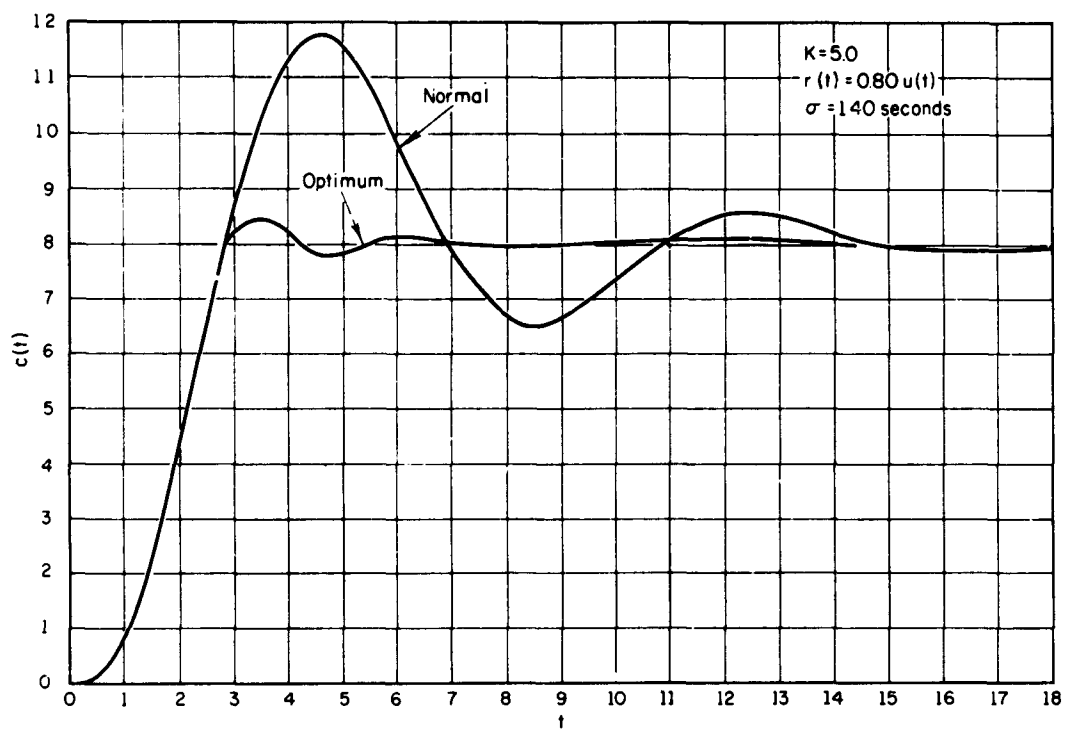


Figure 35. Step Response of Fourth Order System with Complex Roots

$$G(s) = \frac{K}{s(s+3)(s^2 + 2s + 2)}$$

7. SUMMARY, FURTHER PROBLEMS AND CONCLUSIONS

7.1 Summary

A method for the design of a controller to optimize a class of pulse width modulated systems has been presented. The process to be controlled is linear, time-invariant, of arbitrary order and excited by a sequence of pulses generated from information available at arbitrary sampling instants. State variable notation has been used for description of the plant and input processes. Estimation of the state at the sampling instants is provided by linear estimation techniques, accomplishing the identification problem. The information obtained from the filter is then used in the controller to develop a predicted estimate of the future system error which in turn is used to develop the optimum pulse width and sign to minimize a measure of the predicted future system error at each sampling instant. The criterion established, providing the definition of optimality, is based on the minimization of the conditional expectation of a sum-squared performance index. It was shown that the overall system is optimum in this sense. Two alternate schemes for determination of the pulse width and sign were presented allowing flexibility in controller design to include those systems requiring calculation time in the controller comparable to the length of the sampling interval.

The construction of an optimal strategy has been presented in the literature^{10,11,12,13} based on the determination of a finite canonical sequence associated with each initial state which specifies optimal control over the entire transient process during which the state is taken to the origin. Construction of the canonical sequence associated with any arbitrary initial state is determined from the observed value of the state by a reverse time mapping. The phase plane is divided into two regions corresponding to pulse width and relay control in each of these cases. The approach is limited to plants of second order due to the conceptual difficulty of determining the optimal control regions in higher dimensional state spaces.

The technique presented in this report is based on the philosophy of minimizing future system errors at each sampling instant as opposed to the usual reverse time mapping. The future weighted errors are minimized sequentially and the optimal strategy is constructed at each step removing the previous

restriction to second order systems. Input information is also quite general in that random and deterministic phenomena are included as opposed to the previously presented techniques which are restricted to simple deterministic inputs. The usual condition in practice where information about the state is constrained is also included. Sampling may occur at arbitrary times and is not limited to a constant rate. Finally, the application of this technique is no more difficult for a system of arbitrary order than for one of second order. The higher order system requires more computation in the controller but if this restriction is severe an alternate technique has been presented utilizing the length of the sampling interval for computational purposes.

Several systems were simulated on a digital computer using the methods presented in this paper. The results indicate superior performance of the system for ranges of input signal levels with respect to the length of the sampling interval and the gain of the plant process. It was found that input displacement levels of small magnitude corresponded to response time of the system which was smaller than the sampling interval leading to what might be called undesirable results even though the criterion for optimality was satisfied. Also, for input displacement levels of large magnitude, the system was found to be predominantly in the relay mode and hence, pulse width control effort had very little effect.

7.2 Further Problems

The concept of minimizing future errors has led to the technique presented in the preceding work. The future error at only one sampling instant ahead was considered with only a passing mention of the control law which would be applicable if future errors several sample instants ahead were considered. The extension of this technique to include investigation of the predicted error at perhaps two sample instants ahead together with the information about error only one sample ahead and a subsequent decision process based on the desired performance, might prove worthwhile. Various approaches utilizing the predicted errors may provide the insight necessary to achieve excellent performance for systems other than those with pulse width control signals.

No investigation into the stability problem was made in this work. Several interesting results were noted however, when the controller was introduced into the system. The stability boundary determined as a function of sampling time and plant gain for two of the example systems changed appreciably. It was found

that a second order system was stabilized in the optimal mode for both step and ramp inputs for gains twice as large as the maximum gain allowable for stability in the normal mode. A third order system was stabilized in one case by allowing a larger gain than in the normal mode, but on the other hand, the reverse was found true for a different sampling rate. Since the system is non-linear, the interesting stability problem would be extremely difficult to solve but would certainly be a major contribution.

7.3 Conclusions

The techniques presented here are applicable to plants excited by a pulse width modulated signal. The results seem to indicate marked improvement in system performance at only the expense of additional computation which may be accomplished between the sampling intervals. It must be concluded, however, that the science of optimization is still in the infancy stage and the use of predicted future system errors to achieve optimal control is but only an attempt which, hopefully, will help contribute to the maturity of optimization science.

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